White Noise Theory and Its Applications

Takeyuki Hida

1. Preface

White noise does not mean unwanted sounds, but it is useful for science as well as for daily life through the mathematical theory that we shall explain in this note.

The white noise theory is an analysis of functions of *continuously* many variables and is naturally developing into the modern theory of *stochastic analysis*.

It applies to things of all sizes; quantum dynamics, statistical mechanics, biology, statistics, even sociology. Yet, the white noise theory continues to discover new directions in science successfully.

The idea of our theory is *reductionism and analysis* applying for everything that can be modelled as a stochastic system over some probability measure space that we can form. Once a model is given, we form a system of *independent* random variables containing the same information as the stochastic system just modelled. Thus, we are ready to analyse the given system. Then follows mathematical analysis and applications.

One may wonder why it begins with the analysis of functions of "continuously many" variables instead of finite or countably infinite variables. The main reason is that we are interested in a function of analog events: in fact, interesting cases are mostly expressed as functions of continuously many basic random variables. This situation is quite different from the discrete cases where only finite or countably infinite variables are involved.

The main aims of the present note are in order.

i. Rigorous introduction of *noises* in line with reductionism.
ii. Functionals of noises with special emphasis on those of Gaussian case, i.e. white noise. We are naturally led to the space of *generalised functionals* of the noise.
iii. *Calculus* of generalised functionals.
iv. Introduce the infinite dimensional *rotation group* to discuss the harmonic analysis arising from the group.
v. Significant fields of applications.

2. Noises

It is difficult to introduce a system of continuously many independent random variables, e.g. those with parameter space R^1. To fix the idea, assume that they are nontrivial and identically distributed. Then the joint distribution of such a system will not be separable, so that it is impossible to discuss within ordinary theory of calculus. Our first problem is to overcome this difficulty.

Our idea is to take the time derivatives of an additive process, let it be denoted by $Z(t), t \in T$, T being an interval of R^1. We assume that $Z(t)$ is a Lévy process. The $Z(t)$ has independent increments, with separability, so it guarantees that the time derivatives $\dot{Z}(t), t \in T$ define a system of continuously many independent random variables, as is expected.

Now we have to pay a price. Namely, $\dot{Z}(t)$ is no more ordinary random variable, but it is an idealised random variable. Nevertheless, we are happy to have continuously many independent variables.

The $\dot{Z}(t)$ is a generalised stochastic process. Further, to have an analogue of the i.i.d. random variables, we assume that increments of $Z(t)$ in t is stationary. Hence, the $\dot{Z}(t)$ is a stationary idealised process.

Definition 2.1. *A stationary generalised stochastic process $\dot{Z}(t)$ is called a noise.*

We now discuss realisations of $\dot{Z}(t)$, that is, to see the possible kinds of such systems. Appealing to P Lévy's idea, we approximate the noise by a sequence of i.i.d. random variables.

To fix the idea, we consider the time parameter running through $I = [0,1]$. For every n take the division $[k/2^n, (k+1)/2^n]$, $0 \le k \le 2^n - 1$. With each subinterval, we associate i.i.d. random variables X_k^n.

There are three cases. The first two are extremal in the respective ways and are time-dependent. The third one is, as it were a by-product, space-dependent.

i) Assume that each random variable has finite 3rd order moment with mean 0 and variance 2^{-n}. Then, the sum $S_n = \sum_1^{2^n} X_k^n$ converges in law to a standard Gaussian random variable as $n \to \infty$ by the central limit theorem. Also, the law of any consecutive partial sum tends to a Gaussian distribution with variance proportional to the length of the subinterval of I.

Such observations tell us that the probability distribution of a Brownian motion as well as that of white noise can be approximated in a quite natural manner. This is true since the conclusion does not depend on the choice of the probability distributions of the X_k^n.

Thus, we can say that a Brownian motion or a white noise always appears regardless the choice of distribution of X_k^n, so far as we use the trick used above. Note that all the Gaussian distributions are of the same type in distribution. We may therefore say that Gaussian noise is extremal or sitting on top of approximations.

ii) There is another extremal one, sitting not on top, but bottom. It is the Poisson noise.

The setting is the same as above. The probability distribution of X_k^n is chosen so as to be most simple in the sense of probability distribution. Namely, each X_k^n takes only two values, say, 1 and 0 with probability p_n and $1 - p_n$, respectively. This choice is most simple in the sense of the factorisation of distributions. A technical choice is that $p_n = 2^{-n}\lambda$ with some positive constant λ. Then, by the law of small probability, we obtain a Poisson distribution. We can see the similar results for partial sums and obtain a Poisson process $P(t, \lambda)$ with intensity λ. Also we have a Poisson noise $\dot{P}(t, \lambda)$.

Obviously, this is another extremal case, coming from the choice of the distribution of X_k^n.

iii) It is noted that the choice of λ in ii) is quite arbitrary so far as it is positive. It is, in fact, the expectation of the $P(1, \lambda) = P(\lambda)$, where λ is viewed as a space variable.

Now one may ask if there exists a noise depending on the space variable λ. The answer is yes.

Theorem 2.2. *There is a noise depending on the space variable λ.*

The idea of the proof is as follows. We have established the exact form of Poisson noise $\dot{P}(t, \lambda)$ which is viewed as a generalised stochastic process with independent values at every t. With the similar method to ii) and noting the relationship between t and λ (like a duality), we can form space noise denoted by $P'(\lambda)$. It defines a Poisson type distribution.

Discussions made in i), ii) and iii) are useful in decomposition theory of a Lévy process.

3. Generalised Functionals of White Noise

To fix the idea we take, hereafter, a Gaussian noise which is realised by the time derivative $\dot{B}(t)$ of a Brownian motion $B(t), t \in R^1$.

The probability distribution μ of $\dot{B}$ is given on a space of E^* of generalised functions; the dual space of some nuclear space E. The characteristic functional is

$$C(\xi) = e^{-\frac{1}{2}\|\xi\|^2}, \ \xi \in E.$$

There are two ways to define: one is based on polynomials in $\dot{B}(t)$'s, and the other is an infinite dimensional analogue of the Schwartz distribution.

1) First, we have to give a definite position to $\dot{B}(t)$ in a certain space $H_1^{(-1)}$, where the system $\{\dot{B}(t), t \in R^1\}$ is *total*.

For general cases, we must take higher degree polynomials. Since there are continuously many variables, so that general polynomials are of the form

$$\int \cdots \int F(u_1, \cdots, u_n) : \dot{B}(u_1)^{p_1} \cdots \dot{B}(u_n)^{p_n} : (du)^n,$$

where $p = \sum_j p_j$ is the degree of the polynomial and : : means the renormalisation which is done by using idealised Hermite polynomials with variance parameter $\frac{1}{dt}$. Their sums finally span the space $(L^2)^-$ of generalised white noise functionals. The space is a reasonable extension of the space (L^2) involving ordinary white noise functionals with finite variance.

2) This is an infinite dimensional analogue of the Schwartz space $\mathcal{S}$ over R^1. Namely, we use the differential operator

$$D = -\frac{d^2}{du^2} + u^2 + 1$$

and apply the second quantisation method to have the space $(\mathcal{S})$ and its dual space $(\mathcal{S})^*$ which is also called the space of generalised white noise functionals. A significant property is that there is a characterisation of a member in $(S)^*$ (Pothoff–Streit).

The S-transform of generalised functional $\varphi(x)$, x being a sample function of $\dot{B}$, is defined by

$$(S\varphi)(\xi) = C(\xi)\int_{E^*} e^{<x,\xi>}\varphi(x)d\mu(x).$$

Let $\mathbf{F}$ be the image of $(L^2)^-$ under S-transform. We can carry on the analysis smoothly on the space $\mathbf{F}$ in place of $(L^2)^-$ (or $(S)^*$). Useful generalised white noise functionals are often transformed by S to ordinary functionals of ξ.

There are similar advantages for the differential operators. $\partial_t = \frac{\partial}{\partial \dot{B}(t)}$ can be transformed to the Fréchet derivative acting on $\mathbf{F}$. With such helps we can carry on the differential calculus on the space of white noise functionals.

4. Infinite Dimensional Rotation Group

The finite dimensional standard Gaussian distribution is invariant under the rotations around the origin. Such an invariance holds for the white noise measure. A rigorous and convenient definition of the rotations of a nuclear space E, a dense subspace of $L^2(R^1)$, is given as follows.

Definition 4.1. *A continuous linear transformation g acting on E is a rotation of E if the $L^2(R^1)$-norm is kept invariant:*

$$\|g\xi\| = \|\xi\|, \ \ \xi \in E.$$

The collection of rotations forms a group under the ordinary multiplication. The group is denoted by $O(E)$ and is called the rotation group of E. It is topologised under the compact-open topology.

The adjoint g^* of $g \in O(E)$ is defined and the collection of the g^*'s forms a group denoted by $O^*(E^*)$. It is isomorphic to $O(E)$. We can define the operator U_g by

$$(U_g\varphi)(x) = \varphi(g^*x).$$

Theorem 4.2. *For any $g \in O(E)$, the g^* is a measurable transformation on the white noise space (E^*, μ), and U_g is a unitary operator acting on (L^2).*

Proof comes from the fact that the characteristic functional $C(\xi)$ of μ is invariant under $g \in O(E)$.

With these observations we understand that the rotation group provides a characterisation of white noise measure.

Coming to probability theory, we are interested in the roles of the group $O(E)$ in the study of the analysis of white noise functionals, indeed a harmonic analysis arising from $O(E)$. There are subgroups G_n (of $O(E)$) isomorphic to $SO(n)$ and they play the similar roles to the finite dimensional case.

To be more interesting, we can see one-parameter subgroups that come from diffeomorphisms of the parameter space. The simplest and most important subgroup is the one that comes from the time shift. It defines the flow of Brownian motion. It is a good problem to find other one-parameter subgroups describing significant probabilistic properties.

5. Cooperating Fields

There are many cooperating fields not only in mathematics but widely in science and other fields that cooperate with white noise theory. They are not simply applications, but propose interesting problems to be developed in connection with white noise theory. Some examples are in order.

1) Feynman path integrals. In quantum dynamics we consider possible trajectories around the classical path that is determined by Lagrangian. We propose to take a Brownian bridge to express the difference between the classical trajectory and other fluctuating paths. Thus the action integral is a function of white noise, since the velocity is expressed as a function of the time derivative of the Brownian bridge.

Thus the computation of the propagator following Feynman's suggestion turns out to be a white noise calculus. Needless to say, good results have been obtained in the cases where various potentials are given.

2) In statistical mechanics we meet various equations to describe physical phenomena. Some are viewed as nonlinear generalisations of the Langevin equation. There we meet equations involving a noise explicitly. To obtain the explicit solution to the equation we are led to profound analysis of white noise functionals. Since the noise involved there may not be a simple white noise, mathematical theory from wider viewpoints is requested.

3) Molecular biology. We do not wait until observed data with fluctuation. Giving white noise

input to an unknown mechanism like biological organ. Observing the output to see biological functions of the system in question. By using white noise calculus we identify the biological function, where one-parameter subgroup of $O(E)$ is a good tool.

4) We have a characterisation of $(S)^*$-functionals. Once we have a data expressed as a sample function of a stochastic process, we can find if its source is a white noise. We have good examples not yet solved.

5) There are problems in group representation theory related to $O(E)$.

Note. Similar discussions can be done for other noises.

References

[1] T. Hida, *Analysis of Brownian Functionals*, Carleton Univ. Math. Notes 13 (1975).

[2] T. Hida, H.-H. Kuo, J. Potthoff and L. Streit, *White Noise. An Infinite-dimensional Calculus* (Kluwer Academic Publ., 1993).

[3] T. Hida and S. Si, *Lectures on White Noise Functionals* (World Scientific, 2008).

[4] P. Lévy, *Problèmes concrets d'analyse fonctionnelle* (Gauthier-Villars, 1951).

[5] S. Si, *Introduction to Hida Distributions* (World Scientific, 2012).

[6] N. Wiener, *Nonlinear Problems in Random Theory* (MIT Press, 1958).

Takeyuki Hida

Nagoya University and Meijo University, Japan

takeyuki@math.nagoya-u.ac.jp

Takeyuki Hida is currently Professor Emeritus at Nagoya University and Meijo University, Japan. He obtained his BSc from Nagoya University, and PhD from Kyoto University. He has published many research papers and books. His present researches include further developments of white noise theory and its applications to many fields in physics as well as to information theory.