Introduction to Toric Topology

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The n-dimensional *toric variety* is a complex algebraic variety with an algebraic action of $(\mathbb{C}^*)^n$ that has an open dense orbit. For instance, the complex projective space $\mathbb{C}P^n = \mathbb{C}^{n+1} - \{O\}/\sim$ of complex dimension n with an $(\mathbb{C}^*)^n$-action given by

$$(\mathbb{C}^*)^n \times \mathbb{C}P^n \to \mathbb{C}P^n$$
$$(t_1, t_2, \ldots, t_n) \cdot [z_0; z_1; \ldots; z_n] \mapsto [z_0; t_1z_1; \ldots; t_nz_n]$$

is a toric variety since there is an open dense orbit $\{[z_0; z_1; \ldots; z_n] \mid z_i \neq 0 \text{ for all } i\}$. An interesting fact is that such toric varieties can be expressed by a combinatorial method. Among the orbits of action of $(\mathbb{C}^*)^n$ for an n-dimensional toric variety X, let $X_1, \ldots, X_m$ be the closure of orbits with a complex codimension of 1. Here, if you consider the following family of subsets of $[m] = \{1, 2, \ldots, m\}$

$$K_X := \left\{ I \subset [m] \mid \bigcap_{i \in I} X_i \neq \emptyset \right\},$$

you will obtain a simplicial complex. On the other hand, since X_i is fixed by a $\mathbb{C}^*$-subgroup of $(\mathbb{C}^*)^n$, it can be thought that each X_i corresponds to a single element λ_i of $\operatorname{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n) \cong \mathbb{Z}^n$. (In particular, the sign of λ is determined by the direction of the action of $(\mathbb{C}^*)^n$ on the normal bundle of X_i, but here, detailed explanations will be skipped.) Now, let us consider the half lines that spread indefinitely in the direction of λ_i on $\mathbb{R}^n$. If we collect all cones, which are $c_\sigma = \{\sum_{i \in \sigma} c_i \lambda_i \mid c_i \geq 0\}$ for the element σ of K_X, they become combinatorial objects called *fans*. For example, in the above case of $\mathbb{C}P^n$, we can set $X_i = \{[z_0; z_1; \ldots; z_n] \mid z_i = 0\}$, and here,

$$\lambda_i = \begin{cases} \mathbf{e}_i, & \text{if } i = 1, 2, \ldots, n; \\ -\mathbf{e}_1 - \cdots - \mathbf{e}_n, & \text{if } i = n+1 \end{cases}$$

and K_X becomes a simplex. For instance, the fan corresponding to $\mathbb{C}P^2$ is given in Fig. 1.

The details of the determination of the relationship between the toric variety and the fan are given in the books of Fulton [7] or Oda [11]. What is surprising is that when there are toric varieties, the fans can be corresponded to

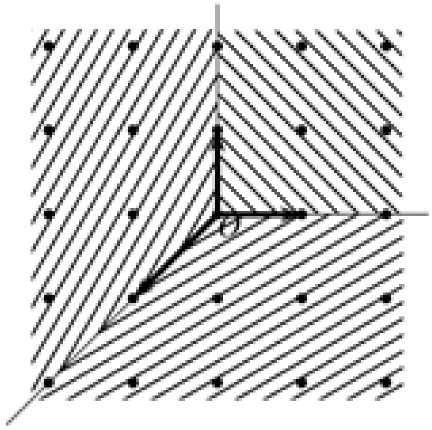

Fig. 1. Fan corresponding to $\mathbb{C}P^2$.

and their reverse is also valid. This is called the fundamental theorem of toric geometry.

1. Fundamental Theorem of Toric Geometry

There is a 1-1 correspondence relationship between the family of toric varieties and the family of fans.

According to the fundamental theorem of toric geometry, we can consider objects of algebraic geometry called toric varieties to be combinatorial objects called fans. This consideration can be the link connecting algebraic geometry and combinatorics. There are many problems that are created from understanding actual combinations in terms of algebraic geometry, or problems that are solved in reversed circumstances. The most famous outcome among them is that the problem regarding the number of faces of simplicial convex polytope proposed by McMullen [9]. In 1971, this problem was solved by Stanley by using the idea of toric geometry. McMullen's problem is stated as follows: If an n-dimensional simplicial convex polytope P is given, let f_i be the number of i-dimensional faces where $(f_0, f_1, \ldots, f_{n-1})$ is called the f-vector of P. The h-vector of P, $(h_0, h_1, \ldots, h_n)$, is defined to satisfy

$$h_0t^n + h_1t^{n-1} + \cdots + h_n = (t-1)^n + f_0(t-1)^{n-1} + \cdots + f_{n-1}.$$

McMullen proposed the following problem, which is nowadays called g-theorem, as the necessary and sufficient conditions for a certain vector $(h_0, h_1, \ldots, h_n)$ to be the h-vector of P.

2. g-theorem

An integral vector $(h_0, h_1, \ldots, h_n)$ (where $h_0 = 1$) is the h-vector of some simplicial convex polytope of dimension n if and only if it satisfies the following conditions:

(1) $h_i = h_{n-i}$ (This is commonly called the Dehn–Sommervile equation.)

(2) $1 = h_0 \le h_1 \le \cdots \le h_{[n/2]}$

(3) $h_{i+1} - h_i \le (h_i - h_{i-1})^{\langle i \rangle}$.

Here, $a^{\langle i \rangle}$ is defined as follows: for positive integers a and i, the natural numbers that satisfy the following equation:

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$$

$a_i > a_{i-1} > \cdots > a_j \ge j \ge 1$ always exist uniquely, where

$$a^{\langle i \rangle} = \binom{a_i + 1}{i + 1} + \binom{a_{i-1} + 1}{i} + \cdots + \binom{a_j + 1}{j + 1}.$$

The existence of the polytope P that has the vector satisfying the necessary and sufficient conditions for this theorem, i.e., the above three conditions (1)–(3) for the h-vector was solved by Billera–Lee [1], and its converse was solved by Stanley [12]. Stanley actively used the theories in various fields including toric geometry, and the proof can be briefly described as follows: In general, if the n-dimensional simplicial convex polytope P exists, there is always a fan (strictly speaking, a simplicial fan) on it, and according to the fundamental theorem of toric geometry, there exists its corresponding projective toric variety X, where it is known that the $2i$-th Betti number $\dim_{\mathbb{Q}} H^{2i}(X; \mathbb{Q})$ of X is the same as h_i of P. Therefore, Property (1) is satisfied by the Poincaré duality theorem of topology, Property (2) by the hard Lefschetz theorem of algebraic geometry, and Property (3) by the result of Macaulay's study using the commutative algebra of $H^*(X)$. Stanleys splendid and neat proof served as a catalyst for integrating toric geometry and combinatorics. (McMullen [10] proved the g-theorem again using a pure combinatorial method.)

Thereafter, the theory of toric geometry was developed by using the methods of topology more frequently instead of using the methods of algebraic geometry. If you think about the unit elements in $\mathbb{C}^*$ of $(\mathbb{C}^*)^n$ acting on a toric variety, you will know that there is also an action of the torus $T^n = (S^1)^n \subset (\mathbb{C}^*)^n$ on the toric variety. Therefore, a toric variety can be understood as a topological space that has the action of the torus. The topological space that contains the torus symmetry has already been studied as an important object in transformation groups, etc., since the 1950s, but its full-scale examination began only after the late 1980s when the ideas of studies on toric varieties were applied to correspond them to combinatorial objects. Pioneering works of Davis–Januszkiewicz [6], Buchstaber–Panov [2], Hattori–Masuda [8], etc., have been expanding the notion of toric varieties to topological spaces that have various torus symmetries. In particular, in such a development stage, the geometric objects frequently used in topological methods including algebraic topological methods were expanded to general topological spaces than the toric varieties. Furthermore, combinatorial objects have been developed not only for fans and simplicial convex polytopes but also in the expanded domains of general objects such as multi-fans or simplicial spheres. Further, these mathematical objects are considered important in symplectic geometry as well since, for example, a symplectic manifold having a Hamiltonian T^n-action can be understood as a toric variety. As such, not only algebraic geometry and combinatorics but also topology and symplectic geometry formed very close relationships with each other.

A series of studies on the topological spaces that have such torus actions are called *toric topology*, and this terminology was first used in the early 1990s by a study group of Manchester University in the UK. Thereafter, Buchstaber–Ray [3] has used the term "Toric Tetrahedron" to express the characteristics of toric topology as an academic subject.

In general, researchers who investigate toric topology are interested in the arbitrary-dimensional topological spaces and are particularly interested in the $2n$-dimensional topological spaces that have the actions of T^n. This is because if there is an action of torus T on the topological space M, at an arbitrary point x of M, $\dim T + \dim T_x \le \dim M$ (where $T_x \subset T$ is the isotropy subgroup at x) must be satisfied.

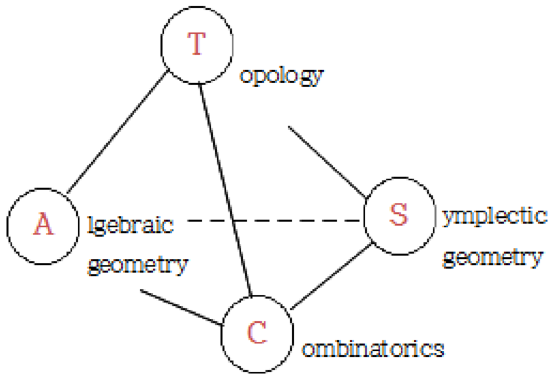

Fig. 2. Toric tetrahedron.

In particular, if this action has a fixed point, $2\dim T \leq M$ must be valid since $\dim T_p = \dim T$ at the fixed point p. That is, a $2n$-dimensional topological space having the action of T^n is a topological space having the most symmetries. Among topological spaces, one usually consider a manifold because it has many nice properties. A *torus manifold* is a closed smooth manifold of even dimension admitting an effective half dimensional torus action with non-empty fixed point set. In mathematics, when an object has many symmetries, it is often expressed as "beautiful". In other words, the studies on such torus manifolds can be viewed as those on the most beautiful objects in topology. Furthermore, since these objects are very natural, many mathematical objects are likely to have structures of torus manifolds. For example, since the surface of the rotating Earth is a two-dimensional manifold S^2 that has an action of S^1 and has fixed points of the North and South Poles, it can be viewed as a torus manifold. Of course, all toric varieties are torus manifolds.

One of the important problems for such torus manifolds is the problem of classifying torus manifolds topologically. Since the basic aim of topology is to classify the topological spaces, the topological classification problem of torus manifolds is an attractive problem itself. Since Euler found an invariant value of the topological space called the Euler number, the algebraic invariants such as the fundamental group and homology have been developed for the purpose of classifying topological spaces. The Poincaré conjecture, which asks whether the topological type of sphere can be characterised by its topological invariants, has been recognised as one of the most important problems in mathematics for a long time. A sphere in the Poincaré conjecture is the one of the simplest manifolds, but as it took the researchers 100 years to solve its topological equivalence, the complete classification of general topological manifolds should be also a very difficult problem — may be nearly impossible to solve. Therefore, we can think of classifying topological spaces having better structures and yet more complex shapes than S^n, and from this perspective, the classification problem of topological spaces having certain geometric structures is important. Therefore, it is very natural to ask about the topological classification of manifolds that have symmetric structures.

One of the interesting questions on the topological classification problem of torus manifolds is "Are smooth compact toric varieties (abbreviated as toric manifolds) classified up to homeomorphism (or diffeomorphism) by their cohomology rings?"

3. Problem (Cohomological Rigidity Problem for Toric Manifolds)

If the cohomology rings of two toric manifolds are isomorphic as graded rings, are these two manifolds homeomorphic or diffeomorphic?

Since many topologists think that cohomology rings are invariants too weak to classify as differential manifolds, they will not ask a question like the above. However, for example, the cohomology rings in Freedman's classification of simply connected four-dimensional manifolds can be sufficiently strong invariants for appropriate types of manifolds. An interesting fact is that no example has been found yet to disprove the above question; in fact, many examples of classifications based on cohomology rings have been discovered. For example, the fact that toric manifolds, where the product of complex projective spaces or the Picard number is 2, are all classified as differential manifolds by cohomology rings was proved by a joint study [4] that I conducted with my academic adviser Prof Dong Youp Suh of KAIST, Korea and Prof Masuda of Osaka City University, Japan.

Toric manifolds not only lead to the above question due to the property of their symmetries but also lead to various types of topological and combinatorial rigidity; thus, reading the paper summary [5] will be helpful in understanding the situations about recent studies.

As discussed above, toric topology is to study mathematical objects that have beautiful symmetries, and because of the good properties of the treated objects, the development of toric topology is progressing rapidly. Reflecting such a trend, a large-scale conference on toric topology will be held in Daejeon as one of the satellite conferences of the ICM 2014 to be held in Seoul; I hope that the readers will show a lot of interest in and support for the same.

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