The Twin Prime Problem and Generalisations

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We give a short introduction to the recent breakthrough theorem of Yitang Zhang that there are infinitely many pairs of distinct primes (p, q) with $|p - q| < 70$ million.

The twin prime problem asks if there are infinitely many primes p such that $p + 2$ is also prime. More generally, one can ask if for any even number a, there are infinitely many primes p such that $p + a$ is also prime. This problem inspired the development of modern sieve theory. Though several sophisticated tools were discovered, the problem defied many attempts to resolve it until recently.

On April 17, 2013, a relatively unknown mathematician from the University of New Hampshire, Yitang Zhang, submitted a paper to the *Annals of Mathematics*. The paper claimed to prove that there are infinitely many pairs of distinct primes (p, q) with $|p - q| < 7 \times 10^7$. This was a major step towards the celebrated twin prime conjecture! A quick glance at the paper convinced the editors that this was not a submission from a crank. The paper was crystal clear and demonstrated a consummate understanding of the latest technical results in analytic number theory. Therefore, the editors promptly sent it to several experts for refereeing. The paper was accepted three weeks later.

In this article, we will outline the proof of this recent breakthrough theorem of Yitang Zhang [1]. Even though this article is only an outline, it should help the serious student to study Zhang's paper in greater detail. An essential ingredient in Zhang's proof is the idea of smoothness which allows him to extend the range of applicability of earlier theorems. (A number is said to be y-smooth if all its prime factors are less than y.) The rudimentary background in analytic number theory is readily obtained from [2] and [3]. This can be followed by a careful study of [4] and the three papers [5–7].

1. Introduction and History

Let $p_1, p_2, \ldots$ be the ascending sequence of prime numbers. The twin prime problem is the question of whether there are infinitely many pairs of primes (p, q) with $|p - q| = 2$. This problem is usually attributed to the ancient Greeks, but this is very much Greek mythology and there is no documentary evidence to support it. The first published reference to this question appeared in 1849 by Alphonse de Polignac who conjectured more generally that for any given even number $2a$, there are infinitely many pairs of primes such that $|p - q| = 2a$.

In a recent paper [1] in the *Annals of Mathematics*, Yitang Zhang proved that there are infinitely many pairs of distinct primes (p, q) with

$$|p - q| < 7 \times 10^7.$$

His proof depends on major milestones of 20th century number theory and algebraic geometry. Thus, it is definitely a 21st century theorem! Undoubtedly, his paper opens the door for further improvements and it is our goal to discuss some of these below.

After de Polignac's conjecture, the first serious paper on the subject was by Viggo Brun in 1915, who, after studying the Eratosthenes sieve, developed a new sieve, now called the Brun sieve, to study twin primes and related questions. He proved that

$$\sum_{p : p+2 \text{ prime}} \frac{1}{p} < \infty.$$

By contrast, the sum of the reciprocals of the primes diverges and so, this result shows that (in some sense) if there are infinitely many twin primes, they are very sparse.

A few years later, in 1923, Hardy and Littlewood [8], made a more precise conjecture on the number of twin primes up to x. They predicted that this number is (see [9, p. 371])

$$\sim 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\frac{x}{\log^2 x}.$$

Here, the symbol $A(x) \sim B(x)$ means that $A(x)/B(x)$ tends to 1 as x tends to infinity.

They used the circle method, originally discovered by Ramanujan and later developed by Hardy and Ramanujan in their research related to the partition function. After Ramanujan's untimely death, it was taken further by Hardy and Littlewood in their series of papers on Waring's problem. (The circle method is also called the "Hardy–Littlewood method" by some mathematicians.) In the third paper of this series, they realised the potential of the circle method to make precise conjectures regarding additive questions, such as the Goldbach conjecture and the twin prime problem.

Based on heuristic reasoning, it is not difficult to see why such a conjecture should be true. The prime number theorem tells us that the number of primes $\pi(x)$, up to x, is asymptotically $x/\log x$. Thus, the probability that a random number in $[1, x]$ is prime is $1/\log x$ and so the probability that both n and $n+2$ are prime is about $1/\log^2 x$. The constant is a bit more delicate to conjecture and is best derived using the theory of Ramanujan–Fourier series expansion of the von Mangoldt function as in a recent paper of Gadiyar and Padma [10] (see also [11] for a nice exposition). However, it is possible to proceed as follows. By the unique factorisation theorem of the natural numbers, we can write

$$\log n = \sum_{d|n}\Lambda(d),$$

where $\Lambda(d) = \log p$ if d is a power of a prime p and zero otherwise. This is called the von Mangoldt function. By the Möbius inversion formula, we have for $n > 1$

$$\Lambda(n) = \sum_{d|n}\mu(d)\log\frac{n}{d} = -\sum_{d|n}\mu(d)\log d,$$

since $\sum_{d|n}\mu(d) = 1$ if $n = 1$ and zero otherwise. Thus, to count twin primes, it is natural to study

$$\sum_{p\le x}\Lambda(p+2),$$

where the sum is over primes p less than x. Using the formula for $\Lambda(n)$, the sum above becomes

$$-\sum_{p\le x}\sum_{d|p+2}\mu(d)\log d = -\sum_{d\le x+h}\mu(d)\log d \sum_{p\le x, p\equiv -2 \pmod d} 1.$$

The innermost sum is the number of primes $p \le x$ that are congruent to $-2 \pmod d$, which for d odd is asymptotic to $\pi(x)/\phi(d)$, where $\pi(x)$ is the number of primes up to x, and ϕ is Euler's function. Ignoring the error terms, our main term now is asymptotic to

$$-\pi(x)\sum_{d\le x+2, d\text{ odd}}\frac{\mu(d)\log d}{\phi(d)}.$$

This suggests that

$$\sum_{p\le x}\Lambda(p+2) \sim \pi(x)\left(-\sum_{d>1, d\text{ odd}}\frac{\mu(d)\log d}{\phi(d)}\right).$$

The infinite series in the brackets is not absolutely convergent. However, it converges conditionally and can be evaluated as follows.

Consider the Dirichlet series

$$F(s) := \sum_{d=1}^{\infty}\frac{\mu(d)}{\phi(d)d^s},$$

which converges absolutely for $\Re(s) > 0$. Now, $F(s)$ admits an Euler product

$$\prod_{p,(p,2)=1}\left(1-\frac{1}{p^s(p-1)}\right)$$

which resembles $1/\zeta(s+1)$ (with the 2-Euler factor removed) and so it is natural to write the product as

$$\zeta(s+1)^{-1}(1-2^{-s-1})^{-1}\prod_{p>2}\left(1-\frac{1}{p^{s+1}}\right)^{-1}\left(1-\frac{1}{p^s(p-1)}\right).$$

It is now easy to see that the Euler product converges absolutely for $\Re(s) \ge 0$ and this gives an analytic continuation of $F(s)$ for $\Re(s) \ge 0$. The twin prime constant is now $F'(0)$ and because $\zeta(s+1)$ has a simple pole at $s = 0$ with residue 1, the term $\zeta(s+1)^{-1}$ has a zero at $s = 0$. Thus, our heuristic reasoning gives

$$\sum_{p\le x}\Lambda(p+2) \sim 2\pi(x)\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right),$$

which agrees with the Hardy–Littlewood conjecture (after applying partial summation). A similar argument provides the conjectured formula of Hardy and Littlewood for the number of prime pairs that differ by an even number $2h$.

The reader will note that one can make the above argument precise by introducing the error terms

$$E(x, d, a) := \pi(x, d, a) - \frac{\pi(x)}{\phi(d)}$$

and it is easy to see that the error term in our calculation is

$$\sum_{d\leq x+2} E(x,d,-2).$$

The Bombieri–Vinogradov theorem states that

$$\sum_{d\leq Q} |E(x,d,-2)| \ll \frac{x}{\log^A x}$$

for any $A > 0$ and $Q \leq x^{1/2}\log^{-B} x$, where $B = B(A)$ is a function of A. In fact, one can take $B(A) = A+5$ (see [3, p. 161]). Elliott and Halberstam [12] have conjectured that the result is valid for any $Q < x^{1-\epsilon}$ for any $\epsilon > 0$. Even admitting this conjecture, we see that the interval $[x^{1-\epsilon}, x]$ still needs to be treated. It is this obstacle that motivates the use of truncated von Mangoldt functions:

$$\Lambda_D(n) := \sum_{d|n, d<D} \mu(d)\log(D/d)$$

and more generally

$$\Lambda_D(n;a) := \frac{1}{a!}\sum_{d|n,d<D} \mu(d)\log^a(D/d)$$

as will be indicated below.

2. The Basic Strategy of Zhang's Proof

Let $\theta(n) = \log n$ if n is prime and zero otherwise. We will use the notation $n \sim x$ to mean that $x < n < 2x$. Now, suppose we can find a positive real-valued function f such that for

$$S_1 = \sum_{n\sim x} f(n),$$

$$S_2 = \sum_{n\sim x} (\theta(n) + \theta(n+2))f(n),$$

we have

$$S_2 - (\log 3x)S_1 > 0,$$

for sufficiently large x. Then we can deduce that there exists an n such that n and $n+2$ are both prime with $x < n < 2x$. Such a technique and a method to choose optimal functions f goes back to the 1950's and is rooted in the Selberg sieve. See for example [2] for a short introduction to the Selberg sieve.

The problem as posed above is intractable. So we generalise the problem and consider sets

$$\mathcal{H} = \{h_1, h_2, \ldots, h_k\}.$$

It is reasonable to expect (under suitable conditions) that there are infinitely many n such that $n+h_1, n+h_2, \ldots, n+h_k$ are all prime. This would be a form of the generalised twin prime problem and was first enunciated in the paper by Hardy and Littlewood alluded to above. Clearly, we need to put some conditions on $\mathcal{H}$. Indeed, if for some prime p the image of $\mathcal{H}$ (mod p) has size p, then all the residue classes are represented by p so that in the sequence,

$$n + h_1, n + h_2, \ldots, n + h_k$$

there will always be some element divisible by p and it is unreasonable to expect that for infinitely many n all of these numbers are prime numbers. So a necessary condition is that $\nu_p(\mathcal{H}) = |\mathcal{H}|$ (mod p) $< p$ for every prime p. Under such a condition, the set is called *admissible* and we expect this to be the only local obstruction.

Zhang [1] proves:

Theorem 1. *Suppose that $\mathcal{H}$ is admissible with $k \geq 3.5 \times 10^6$. Then, there are infinitely many positive integers n such that the set*

$$\{n + h_1, \ldots, n + h_k\}$$

contains at least two primes. Consequently,

$$\liminf_{n\to\infty} (p_{n+1} - p_n) < 7 \times 10^7.$$

In other words, $p_{n+1} - p_n$ is bounded by 7×10^7 for infinitely many n.

Zhang shows that the second assertion follows from the first if we choose for $\mathcal{H}$ a set of $k_0 = 3.5 \times 10^6$ primes lying in the interval $[3.5 \times 10^6, 7 \times 10^7]$. This can be done since

$$\pi(7 \times 10^7) - \pi(3.5 \times 10^6) > 3.5 \times 10^6$$

from known explicit upper and lower bounds for $\pi(x)$ due to Dusart [13]. That such a set of primes is admissible is easily checked. Indeed, if $p > k_0$, $\nu_p(\mathcal{H}) \leq k_0 < p$. If $p < k_0$ and $\nu_p(\mathcal{H}) = p$, then one of the prime elements is divisible by p and hence equal to p, a contradiction since we chose elements of $\mathcal{H}$ to be primes $> k_0$.

The main strategy of the proof goes back to the paper by Goldston, Pintz and Yildirim [4] where they consider

$$S_1 = \sum_{n\sim x} f(n),$$

$$S_2 = \sum_{n\sim x} \left(\sum_{h\in\mathcal{H}} \theta(n+h)\right) f(n).$$

The idea is to show that for some admissible $\mathcal{H}$, we have

$$S_2 - (\log 3x)S_1 > 0.$$

This would imply that there are at least two primes among the sequence

$$n + h_1, \ldots, n + h_k.$$

They choose, $f(n) = \lambda(n)^2$ with

$$\lambda(n) = \frac{1}{(k+\ell)!} \sum_{d|P(n), d<D} \mu(d) g(d),$$

where μ denotes the familiar Möbius function and

$$g(d) = \left(\log \frac{D}{d}\right)^{k+\ell},$$

and

$$P(n) = \prod_{h \in \mathcal{H}} (n+h).$$

What is now needed is a good *upper bound* for S_1 and a good *lower bound* for S_2. This is the same strategy adopted in [4]. To elaborate, let $C_i(d)$ be the set of solutions (mod d) for $P(n - h_i) \equiv 0$ (mod d) and define the *singular series* $\mathfrak{S}$ by

$$\mathfrak{S} = \prod_p \left(1 - \frac{\nu_p(\mathcal{H})}{p}\right)\left(1 - \frac{1}{p}\right)^{-k}.$$

With

$$T_1^* = \frac{1}{(k+2\ell)!}\binom{2\ell}{\ell}\mathfrak{S}(\log D)^{k+2\ell} + o((\log x)^{k+2\ell})$$

and

$$T_2^* = \frac{1}{(k+2\ell+1)!}\binom{2\ell+2}{\ell+1}\mathfrak{S}(\log D)^{k+2\ell+1} + o((\log x)^{k+2\ell+1}),$$

the argument of [4] leads to

$$S_2 - (\log 3x)S_1 = (kT_2^* - (\log x)T_1^*)x + O(x(\log x)^{k+\ell}) + O(E),$$

where

$$E = \sum_{1\le i\le k} \sum_{d<D^2} \mu(d)\tau_3(d)\tau_{k-1}(d) \sum_{c \in C_i(d)} \Delta(\theta; d, c)$$

and

$$\Delta(\theta; d, c) = \sum_{n\sim x, n\equiv c (\mathrm{mod}\, d)} \theta(n) - \frac{1}{\phi(d)} \sum_{n\sim x} \theta(n).$$

We write $a(x) = o(b(x))$ if $a(x)/b(x)$ tends to zero as x tends to infinity. We also write for non-negative $b(x)$, $a(x) = O(b(x))$ (or $a(x) \ll b(x)$) if there is a constant K such that $|a(x)| \le Kb(x)$ for all x.

Let us look at the main term. A quick calculation shows that it is

$$\left(\frac{2k(2\ell+1)}{(k+2\ell+1)(\ell+1)}\log D - \log x\right) \times \frac{(\log D)^{k+2\ell}}{(k+2\ell)!}\binom{2\ell}{\ell}\mathfrak{S}x.$$

We need to choose D so that the term in brackets is positive. Let $D = x^\alpha$. The term in brackets is positive provided

$$\frac{2k(2\ell+1)}{(k+2\ell+1)(\ell+1)}\alpha - 1 > 0.$$

That is, we need

$$\alpha > \frac{(k+2\ell+1)(\ell+1)}{2k(2\ell+1)} = \frac{1}{4}\left(1 + \frac{2\ell+1}{k}\right)\left(1 + \frac{1}{2\ell+1}\right).$$

From this, we see that if k and ℓ are chosen to be sufficiently large and ℓ/k is sufficiently small, the quantity on the right side is asymptotic to $1/4$. Thus, if we can choose $\alpha > 1/4$ then we can find choices of k and ℓ for which the main term is positive.

The error term is easily recognised to be related to the Bombieri–Vinogradov theorem which shows that for any $\alpha < 1/4$, the error is negligible. So we seem to be at an impasse. However, a well-known conjecture of Elliott and Halbertsam [12] predicts that the error is negligible for *any* $\alpha < 1/2$. This is where things stood in 2005 after the appearance of the paper [4].

The new contribution of Zhang is that in the sums T_1^* and T_2^* (which are actually defined as terms involving the Möbius function and $g(d)$), he notes that terms with divisors d having a large prime divisor are relatively small. So if we let P be the product of primes less than a small power of x and impose the condition that $d|P$ in the Bombieri–Vinogradov theorem then he is able to establish the following:

Theorem 2. *For $1 \le i \le k$, we have*

$$\sum_{d<D^2, d|P} \sum_{c \in C_i(d)} |\Delta(\theta; d, c)| \ll \frac{x}{\log^A x},$$

for any $A > 0$ and $D = x^\alpha$ with $\alpha = 1/4 + 1/1168$.

This theorem is the new innovation. Zhang admits that his choice of k may not be optimal and that the optimal value of k is "an open problem that will not be discussed in this paper".

After the appearance of Zhang's paper, several blogs have discussed improvements, the most notable being Tao's blog [14] and another blog [15], where (as of July 10, 2013), k in Zhang's theorem has been reduced to 1466 and the gap between consecutive primes is now at most 12,006 infinitely often. These are encouraging developments and perhaps we are well on our way to resolving the twin prime conjecture in the foreseeable future.

3. A Closer Look at S_1 and S_2

The analysis of S_1 and S_2 follows closely the treatment in [4] but with a small change. Zhang [1] observes that it is convenient to introduce the condition $d|\mathcal{P}$ with $\mathcal{P}$ being the product of primes less than x^{ϖ} with $\varpi = 1/1168$. With this understanding, the terms S_1 and S_2 are easily handled by direct expansion of the square. Indeed, the first term for S_1 is

$$\sum_{\substack{q\le D\\ q|\mathcal{P}}}\sum_{\substack{r\le D\\ r|\mathcal{P}}}\mu(q)\mu(r)g(q)g(q)\sum_{\substack{n\sim x\\ [q,r]|P(n)}}1.$$

Following Zhang, let $\varrho_1(q)$ be the number of solutions of the congruence $P(n)\equiv 0 \pmod q$ for q squarefree and zero otherwise. By the Chinese remainder theorem, this is a multiplicative function and we have $\varrho_1(p) = k$ if p is coprime to

$$\prod_{1\le i<j\le k}|h_i - h_j|$$

and in general, $\varrho_1(p)\le k$. Thus, the innermost sum is

$$x\frac{\varrho_1([q,r])}{[q,r]} + O(\varrho_1([q,r])),$$

where the implied constant is bounded by unity. The main term of S_1 is

$$x\sum_{\substack{q\le D\\ q|\mathcal{P}}}\sum_{\substack{r\le D\\ r|\mathcal{P}}}\mu(q)\mu(r)g(q)g(r)\frac{\varrho([q,r])}{[q,r]} + O(D^{2+\epsilon}),$$

for any $\epsilon > 0$ since $\varrho([q,r]) \le k^{\omega([q,r])}$, where $\omega(n)$ denotes the number of distinct prime factors of n. Elementary number theory shows that this function is $O(D^{\epsilon})$ for $q,r\le D$ and any $\epsilon>0$.

We let $d_0 = (q,r)$ and write $q = d_0d_1$, $r = d_0d_2$ with $(d_1,d_2) = 1$. The sum S_1 now becomes

$$T_1x + O(D^{2+\epsilon}),$$

where

$$T_1 = \sum_{d_0|\mathcal{P}}\sum_{d_1|\mathcal{P}}\sum_{d_2|\mathcal{P}}\frac{\mu(d_1)\mu(d_2)\varrho_1(d_0d_1d_2)}{d_0d_1d_2}g(d_0d_1)g(d_0d_2)\,.$$

The point is that the same sum without the restriction $d_i|\mathcal{P}$ for $i = 0,1,2$ has already been studied in [4] and the initial section of Zhang's paper is devoted to showing that (essentially) the same asymptotic formula of [4] (namely the formula for T_1^* given above) still holds with the extra condition $d|\mathcal{P}$. (More precisely, what is derived is an upper bound for T_1 which is within e^{-1200} of T_1^*.)

To derive a lower bound for S_2, we have (after a minor change of variables)

$$S_2 = \sum_{i=1}^{k}\sum_{n\sim x}\theta(n)\lambda(n-h_i)^2.$$

We expand the square, interchange summation and obtain

$$\sum_{i=1}^{k}\sum_{q|\mathcal{P}}\sum_{r|\mathcal{P}}\mu(q)\mu(r)g(q)g(r)\sum_{\substack{n\sim x\\ [q,r]|P(n-h_i)}}\theta(n).$$

To handle the innermost sum, we observe that the condition

$$P(n-h_i)\equiv 0 (\operatorname{mod} d)\quad (n,d) = 1$$

is equivalent to $n\equiv c \pmod d$ for some $c\in C_i(d)$. For $d = p$, a prime, this number is the number of distinct residue classes (mod p) occupied by the set $\{h_i - h_j : h_i \not\equiv h_j (\operatorname{mod} p)\}$ which is $\nu_p(\mathcal{H}) - 1$. Thus, defining a multiplicative function (supported only on squarefree values of d) $\varrho_2(d)$ by setting $\varrho_2(p) = \nu_p(\mathcal{H}) - 1$ and extending it multiplicatively, we obtain (using notation introduced earlier), the innermost sum as

$$\sum_{c\in C_i([q,r])}\sum_{\substack{n\sim x\\ n\equiv c(\operatorname{mod}[q,r])}}\theta(n) = \frac{\varrho_2([q,r])}{\phi([q,r])}\sum_{n\sim x}\theta(n) + \sum_{c\in C_i([q,r])}\Delta(\theta;[q,r],c).$$

Now it is an elementary exercise (see for example, [2]) to show that the number of pairs $\{q,r\}$ such that $[q,r] = d$ is given by the divisor function $\tau_3(d)$ which is the number of ways of writing d as a product of three positive integers. Thus, we can simplify this to

$$\sum_{n\sim x}\theta(n)\lambda(n-h_i)^2 = T_2\sum_{n\sim x}\theta(n) + O(E_i),$$

where

$$T_2 = \sum_{q|\mathcal{P}}\sum_{r|\mathcal{P}}\frac{\mu(q)g(q)\mu(r)g(r)}{\phi([q,r])}\varrho_2([q,r])$$

and

$$E_i = \sum_{d<D^2, d|\mathcal{P}}\tau_3(d)\varrho_2(d)\sum_{c\in C_i(d)}|\Delta(\theta,d,c)|.$$

The error term is estimated using Theorem 2. Indeed, by the Cauchy–Schwarz inequality

$$E_i \ll \left(\sum_{d<D^2}\sum_{c\in C_i(d)}\tau_3^2(d)\varrho_2^2(d)|\Delta(\theta;d,c)|\right)^{1/2} \times\left(\sum_{d<D^2, d|\mathcal{P}}\sum_{c\in C_i(d)}|\Delta(\theta;d,c)|\right)^{1/2}.$$

On the first factor, we use the trivial estimate

$$|\Delta(\theta; d, c)| \ll \frac{x}{d} + 1$$

and see (by elementary number theory) that its contribution is at most a power of a logarithm. For the second factor, we use Theorem 2 to save the few powers of logarithm. Thus,

$$S_2 = kT_2 x + O(x(\log x)^{-A}).$$

As before, we can rewrite T_2 as

$$T_2 = \sum_{d_0|\mathcal{P}} \sum_{d_1|\mathcal{P}} \sum_{d_2|\mathcal{P}} \frac{\mu(d_1 d_2)\varrho_2(d_0 d_1 d_2)}{\phi(d_0 d_1 d_2)} g(d_0 d_1) g(d_0 d_2).$$

Again, this sum without the restriction $d_i|\mathcal{P}$ $i = 0, 1, 2$ was treated in [4] and shown to have the asymptotic behaviour given by T_2^*. Zhang shows that T_2 does not differ much from T_2^* (more precisely, that it is within a factor of $e^{-1181.579}$ of T_2^*). Thus,

$$S_2 - (\log 3x) S_1 \geq \omega \mathfrak{S} x (\log D)^{k+2\ell+1} + o(x(\log x)^{k+2\ell+1}),$$

where

$$\omega = \frac{1}{(k+2\ell)!}\binom{2\ell}{\ell}\left(\frac{2(2\ell+1)k}{(\ell+1)(k+2\ell+1)} - \frac{1}{\alpha}\right)$$

nearly. For $\alpha = 1/4 + 1/1168$, it is easily verified that $\omega > 0$.

This calculation already shows that if we have the Elliott–Halberstam conjecture, then for some k_0 and any admissible k_0-tuple $\mathcal{H}$, the set $\{n + h : h \in \mathcal{H}\}$ contains at least two primes for infinitely many values of n. (Tao has labelled this as DHL[k_0, 2].) Farkas, Pintz and Revesz [16] made this relationship a bit more precise as follows. Suppose we have for any $A > 0$,

$$\sum_{d<Q} \max_{(a,d)=1} |E(x, d, a)| \ll \frac{x}{\log^A x},$$

with $Q = x^\theta$. We call this the modified Elliott–Halberstam conjecture EH[θ]. Let j_n denote the first positive zero of the Bessel function (of the first kind) $J_n(x)$ given by the power series

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{2^{2j+n} j!(n+j)!} x^{2j+n}.$$

Then we may take any $k_0 \geq 2$ which satisfies the inequality

$$\frac{j_{k_0-2}^2}{k_0(k_0-1)} < 2\theta$$

and for this k_0 we have DHL[k_0, 2]. The left-hand side is greater than 1 and tends to 1 as k_0 tends to infinity.

4. Variations of Bombieri–Vinogradov Theorem and Extensions

Since we are unable to prove EH[θ] for any $\theta > 1/2$, we look for some suitable modification. Based on the works of Motohashi, Pintz and Zhang, Tao [14] makes the following conjecture which he labels as MPZ[ϖ, δ]: let $\mathcal{H}$ be a fixed k_0-tuple (not necessarily admissible) with $k_0 \geq 2$. Fix w and set W to be the product of the primes less than w. Let b mod W be a coprime residue class and put $I = (w, x^\delta)$. Let S_I be the set of squarefree numbers all of whose prime factors lie in I. Put

$$\Delta_{b,W}(\Lambda; q, a) := \sum_{\substack{n\sim x, n\equiv a (\bmod q)\\ n\equiv b(\bmod W)}} \Lambda(n) - \frac{1}{\phi(q)} \sum_{n\sim x, n\equiv b(\bmod W)} \Lambda(n)$$

and $C(q)$ is the set of zeros (mod q) of the polynomial $P(n)$. Then the conjecture MPZ[ϖ, δ] is that

$$\sum_{\substack{q<x^{\frac{1}{2}+\varpi}\\ q\in S_I}} \sum_{a\in C(q)} |\Delta_{b,W}(\Lambda; q, a)| \ll \frac{x}{\log^A x}$$

for any fixed $A > 0$. Zhang proved that MPZ[ϖ, ϖ] holds for any $0 < \varpi < 1/1168$. Apparently, Zhang's argument can be extended to show that MPZ[ϖ, δ] is true provided

$$207\varpi + 43\delta < \frac{1}{4}.$$

The relationship of the MPZ conjecture to the DHL conjecture is given by (see [14]) the following result. Let $0 < \varpi < 1/4$ and $0 < \delta < 1/4 + \varpi$. Let $k_0 \geq 2$ be an integer which satisfies

$$1 + 4\varpi > \frac{j_{k_0-2}^2}{k_0(k_0-1)}(1+\kappa),$$

where

$$\kappa := \sum_{1\leq n<\frac{1+4\varpi}{2\delta}} \left(1 - \frac{2n\delta}{1+4\varpi}\right)^{k_0/2} \prod_{j=1}^{n}\left(1 + 3k_0 \log\left(1 + \frac{1}{j}\right)\right).$$

Then MPZ[ϖ, δ] implies DHL[k_0, 2]. It is the fine tuning of this theorem along with other observations (regarding admissible sets) that have led to the numerical improvements in Zhang's theorem.

Thus, to prove MPZ[ϖ, δ], we may restrict our moduli to be in the range $(x^{1/2-\epsilon}, x^{1/2+2\varpi})$ since the initial range $(1, x^{1/2-\epsilon})$ can be treated using the classical Bombieri–Vinogradov theorem. Also, it is not difficult to see that θ can be replaced by the von Mangoldt function Λ since the contribution from prime powers (squares and higher) can be shown to be negligible.

An important idea in all proofs of Bombieri–Vinogradov theorem is to decompose the von Mangoldt function into sums of "short sums". To be precise, let us define the Dirichlet convolution of two arithmetic functions f, g to be

$$(f * g)(n) := \sum_{d|n} f(d)g(n/d).$$

Let $L(n) = \log n$, $1(n) = 1$ for all n and set $\delta(n) = 1$ if $n = 1$ and zero otherwise. Then,

$$\Lambda = \mu * L \quad \text{and} \quad \delta = \mu * 1.$$

If we write f^{*n} to denote the n-fold Dirichlet convolution, then

$$\Lambda = \mu^{*10} * 1^{*9} * L$$

is a fact utilised by Zhang (in his Lemma 6) to decompose the von Mangoldt function into "short sums". Let $x^* > (2x)^{1/10}$ and write $\mu = \mu_{\le x^*} + \mu_{>x^*}$, where in the first term, μ is restricted to $[1, x^*]$ and in the second, to the range $> x^*$. Clearly

$$\mu_{>x^*}^{*10} * 1^{*9} * L = 0,$$

since $n \sim x$ cannot be factored as a product of 10 terms each larger than x^*. Thus, writing $\mu_{>x^*} = \mu - \mu_{\le x^*}$ and using the binomial formula we see easily that

$$\Lambda = \sum_{j=1}^{10} (-1)^{j-1} \binom{10}{j} \mu_{\le x^*}^{*j} * 1^{*(j-1)} * L$$

which is an identity of Heath-Brown (but the idea of decomposing arithmetic functions in this way goes back to Linnik).

One can also use the formal identity

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{\zeta'(s)}{1 + (\zeta(s) - 1)}$$
$$= \zeta'(s)(-1 + (\zeta(s) - 1) - (\zeta(s) - 1)^2 + \cdots).$$

This allows one to write the von Mangoldt function as a sum of divisor functions and in this way one reduces the study of primes in arithmetic progressions to the study of divisor sums in arithmetic progressions.

In any case, one uses this decomposition of Λ in the treatment of d in the range $[x^{1/2-\epsilon}, x^{1/2+2\varpi}]$. The decomposition leads to three kinds of sums (called Type I, II and III in the literature, not to be confused with the types occurring in the Vaughan method). The first type involves convolutions $\alpha * \beta$, where β is supported on the interval $[x^{3/8+8\varpi}, x^{1/2-4\varpi}]$ which forces the argument of α to be in $[x^{1/2+4\varpi}, x^{5/8-8\varpi}]$. Type II sums again involve convolutions of the form $\alpha * \beta$, but with β now supported on $[x^{1/2-4\varpi}, x^{1/2}]$ so that α is supported on $[x^{1/2}, x^{1/2+4\varpi}]$, and Type III are the remaining types.

In 1976, Motohashi [17] derived a general induction principle to derive theorems of Bombieri–Vinogradov type for a wide class of arithmetical functions. Much of the treatment of these types of sums follows earlier work of Bombieri, Friedlander and Iwaniec [5] and one needs to verify that the estimates are still valid with the extra condition $d|P$. The point to note is that in the range under consideration, namely $d > x^{1/2-\epsilon}$, the condition that $d|P$ means we can factor d as $d = rq$ with r lying in a suitable interval. This factorization turns out to be crucial in the estimates. Thus, "smoothness" of d is essential in this part of the argument.

Another noteworthy point involves Zhang's estimation of type III sums. His analysis leads to the question of estimating hyper-Kloosterman sums for which Bombieri and Birch have given estimates using Deligne's work on the Weil conjectures. Therefore, this work on the twin prime problem is very much a 21st century theorem!

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