Problem Corner 7

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Welcome to Problem Corner 7. If you solve any of these problems, please send your solutions to APMN@wspc.com by 1 February 2018. A *book prize* will be awarded to the person with the best submission. The solutions will be posted in a future Problem Corner.

Problem 1 — Minimal Die

You would like to construct a six-sided die with the following properties:

- Each face contains a distinct positive integer.
- The integers on two neighbouring faces should differ by at least two.

What is the minimal sum of the six numbers on the die?

Problem 2 — Random Cuts

Two points are chosen uniformly at random along the length of a long stick. The stick is then cut at these two points, resulting in three pieces.

What is the probability that the three lengths can form a triangle?

Problem 3 — Fibonacci Divisibility

Define the Fibonacci sequence by $F_1 = F_2 = 1$ and

$$F_{n+2} = F_{n+1} + F_n, \quad n \geq 1.$$

1) For each positive integer k, prove that there exists some positive integer n such that F_n is divisible by k.
2) For all positive integers a and b, prove that F_a is divisible by F_b if and only if a is divisible by b.

Problem 4 — Posting Boxes

At a special post office, the cost of posting a rectangular box is equal to the sum of its length, width and height. Suppose that box A can fit inside box B completely.

Prove that it is cheaper to post A than B.

Problem Corner 6.2 Solutions

Problem 1 — Magic Square

Nine positive integers are placed in a 3×3 array, such that the three columns, the three rows and the two main diagonals all have the same product. Let x be the middle integer of the top row, and y be the leftmost integer of the middle row.

Prove that xy is a perfect square.

Solution: For convenience, let us use the following table as reference.

a	x	c
y	d	e
f	g	h

Let the common product of each row, column and main diagonal be P. By considering the left column, middle column, bottom row and the diagonal starting at top left, we must have

$$ayf = xdg = fgh = adh = P.$$

Multiplying the four products together, we have

$$(adfgh)^2 xy = P^4.$$

Since every variable is a positive integer, it follows immediately that xy must be a perfect square.

Problem 2 — Club Members

In a school there are n students and some number of clubs. Each club has an odd number of members. Furthermore, each pair of clubs has an even number of members in common.

What is the maximum possible number of clubs in the school?

Solution: We claim that the maximum number of clubs is n. This value can be achieved by simply forming n disjoint clubs, each containing a single student. So each club has one member, while each pair of clubs have zero members in common. This satisfies the conditions of the problem.

It remains to show that it is not possible to have more than n clubs. For the sake of contradiction, suppose at least $n + 1$ clubs are possible. For each club i, let C_i be a vector of length n such that the j-th entry of C_i is either 1 if student j is in club i, or 0 if student j is not in club i. The key idea is to consider $C_1, \ldots, C_{n+1}$ as elements of the vector space $\mathbb{F}_2^n$, in other words, vectors whose entries are taken in modulo 2.

According to the condition of the problem, C_i has an odd number of 1's, while C_i and C_j has an even number of 1's in common. These conditions can be written in terms of the following dot products

$$C_i \cdot C_i = 1, \quad C_i \cdot C_j = 0, \ i \neq j.$$

Since we have $n+1$ vectors in an n-dimensional vector space, they must be linearly dependent. In other words, there exists a linear combination

$$\sum_{i=1}^{n+1} a_i C_i = 0$$

such that not all of the a_i's are zeros. However, by taking the dot product of the equation and C_k, we see that, for each k,

$$0 = 0 \cdot C_k = \sum_{i=1}^{n+1} a_i C_i \cdot C_k$$
$$= a_k(1) + \sum_{i \neq k} a_i(0) = a_k.$$

This contradicts the linear dependence property. Therefore the maximum number of clubs is n.

Problem 3 — Polynomial Equation

Find all polynomials $p(x)$ with real coefficients such that

$$p(x)p(x+1) = p(x^2)$$

is satisfied for all real numbers x.

Solution: We first deal with the case of the constant polynomial $p(x) = c$. Immediately we have $c^2 = c$, which implies that $c = 0$ or 1.

In the case of a non-constant polynomial, by the fundamental theorem of algebra, there exists at least one complex root. Let $z \in \mathbb{C}$ be any root of p. Substituting $x = z$, we have

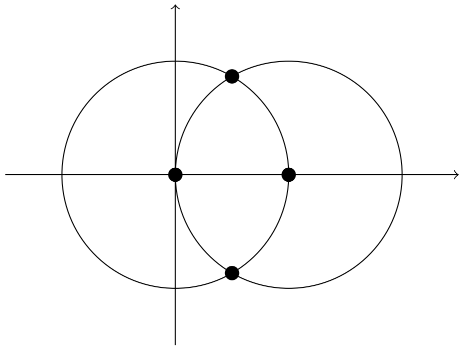
$$0 = p(z)p(z+1) = p(z^2).$$

Hence z^2 is also a root of p. Applying this iteratively, we see that every member of the set $\{z^{2^n}, n \geq 0\}$ is a root of p. If $|z| > 1$, the elements of $\{z^{2^n}, n \geq 0\}$ are all distinct since their moduli are strictly increasing. This is a contradiction since p can only have finitely many roots. A similar contradiction is reached for $0 < |z| < 1$. Thus every root z of p must satisfy $|z| = 0$ or 1.

Next, let us substitute $x = z - 1$ into the condition, we have

$$0 = p(z-1)p(z) = p((z-1)^2).$$

Hence $(z-1)^2$ is also a root, which implies that $|z-1| = 0$ or 1. Recall that we have already established $|z| = 0$ or 1. As shown by the following diagram, these two conditions imply that all roots of p must belong to the set $S = \{0, 1, e^{i\pi/3}, e^{-i\pi/3}\}$.



But recall that z^2 is also a root, this quickly eliminates $e^{i\pi/3}$ and $e^{-i\pi/3}$ since their squares do not belong to S. So the only possible roots are 0 and 1. Hence the polynomial must be of the form

$$p(x) = cx^a(x-1)^b, \quad c \neq 0.$$

Substituting this back into the condition, we have

$$c^2 x^{a+b}(x-1)^b(x+1)^a = cx^{2a}(x^2-1)^b.$$

By comparing the leading term, we see that $c = 1$. Further cancellation yields

$$x^b(x+1)^{a-b} = x^a, \quad \Longrightarrow \quad \left(\frac{x+1}{x}\right)^{a-b} = 1.$$

Since this holds for all $x \neq 0$, we must have $a = b$.

Therefore the possible polynomials are

$$p(x) = 0, \ p(x) = 1 \text{ and } p(x) = x^a(x-1)^a, \ a \geq 1.$$

It is straightforward to verify that all of them are indeed solutions.

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