Problem Corner 6.2

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Welcome to Problem Corner 6.2. If you solve any of these problems, please send your solutions to APMN@wspc.com by 1 February 2017. A *book prize* will be awarded to the person with the best submission. The solutions will be posted in a future Problem Corner.

Problem 1 — Magic Square

Nine positive integers are placed in a 3×3 array, such that the three columns, the three rows and the two main diagonals all have the same product. Let x be the middle integer of the top row, and y be the leftmost integer of the middle row.

Prove that xy is a perfect square.

Problem 2 — Club Members

In a school there are n students and some number of clubs. Each club has an odd number of members. Furthermore, each pair of clubs has an even number of members in common.

What is the maximum possible number of clubs in the school?

Problem 3 — Polynomial Equation

Find all polynomials $p(x)$ with real coefficients such that

$$p(x)p(x+1) = p(x^2)$$

is satisfied for all real numbers x.

Problem Corner 6.1 Solutions

Problem 1 — Right Angles

Let k be the number of interior right angles in an n-sided polygon. Note that a $270°$ interior angle does not count as an interior right angle.

Prove that

$$k \le \frac{2}{3}(n+2).$$

Solution: Aside from the k interior right angles, let us label the other $n-k$ angles by $\theta_1, \ldots, \theta_{n-k}$. By definition, each angle must satisfy $\theta_i < 360°$ for $i = 1, \ldots, n-k$.

It is well known that the interior angle sum of an n-sided polygon is $(n-2)\times 180°$. Thus we have

$$\begin{aligned}(n-2)\times 180° &= k \times 90° + \sum_{i=1}^{n-k} \theta_i \\ &\le k \times 90° + (n-k)\times 360° \\ &= n \times 360° - k \times 270°.\end{aligned}$$

This implies

$$k \times 270° \le n \times 180° + 360°,$$

which simplifies to

$$k \le \frac{2}{3}n + \frac{4}{3},$$

as required.

Problem 2 — Inequality

Prove that if x, y, z are distinct positive real numbers, then

$$\frac{x(y-z)}{y+z} + \frac{y(z-x)}{z+x} + \frac{z(x-y)}{x+y} \ne 0.$$

Solution: For the sake of contradiction, let us assume that equality is possible. By multiplying everything by the denominators, we get the following equation

$$\begin{aligned}&x(y-z)(z+x)(x+y) + y(z-x)(x+y)(y+z) \\ &\quad + z(x-y)(y+z)(z+x) = 0. \qquad (1)\end{aligned}$$

The left side of (1) is a degree 4 symmetric polynomial in x, y and z. We will show that it can actually be factorised.

By substituting $x = y$ into (1), we have the equality

$$2x^2(x-z)(z+x) + 2x^2(z-x)(x+z) + 0 = 0.$$

This means that $(x-y)$ is a factor of the polynomial. By symmetry, $(y-z)$ and $(z-x)$ are also factors. Hence our polynomial must be

$$\begin{aligned}&x(y-z)(z+x)(x+y) + y(z-x)(x+y)(y+z) \\ &\quad + z(x-y)(y+z)(z+x) \\ &= (x-y)(y-z)(z-x)P(x,y,z), \qquad (2)\end{aligned}$$

where $P(x,y,z)$ is a degree 1 symmetric polynomial. In other words, it must have the form of

$$P(x,y,z) = a(x+y+z) + b.$$

By noting that the left side of (2) only has degree 4 terms, we can conclude that $b = 0$ and $a \neq 0$. Thus we have

$$a(x-y)(y-z)(z-x)(x+y+z) = 0.$$

But since x, y and z are distinct positive real numbers, this is a contradiction. Therefore the equality in (1) was not possible in the first place, completing the solution.

Problem 3 — Random Numbers

A computer is generating random numbers uniformly chosen from the interval $[0, 1]$. This process continues until the sum of all numbers generated is at least 1.

On average, how many numbers are required to reach a sum of at least 1?

Solution: Let $x \in (0, 1]$ be a real number and let $f(x)$ be the expected number of random drawing needed to reach a sum of at least x. For small values of x, it is easy to see that $f(x)$ will be slightly above 1. So for convenience, let $f(0) = 1$. We will solve the problem by finding f.

Let the first random number be R. If $R \geq x$, then the sum has exceeded x so one drawing is sufficient. The probability of this occurring is $1-x$. Otherwise, we will need to exceed a sum of $x - R$ from the second drawing onwards. Hence we have the equation

$$\begin{aligned} f(x) &= 1 \times \mathbb{P}(R \geq x) + \mathbb{E}(1 + f(x-R)) \,|\, R < x) \\ &= (1-x) + \int_0^x 1 + f(x-r)\,dr \\ &= 1 + \int_0^x f(x-r)\,dr \\ &= 1 + \int_0^x f(u)\,du. \end{aligned} \tag{3}$$

Differentiating both sides of (3) with respect to x and applying the fundamental theorem of calculus, we have the following straightforward differential equation,

$$\frac{d}{dx} f(x) = f(x), \quad f(0) = 1.$$

So the solution is $f(x) = e^x$. Therefore the expected number of drawings required to reach a sum of at least 1 is given by $e \approx 2.718$.

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