The Farey Structure of the Gaussian Integers

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Arrange three circles so that every pair is mutually tangent. Is it possible to add another tangent to all three? The answer, as described by Apollonius of Perga in Hellenistic Greece, is yes, and, indeed, there are exactly two solutions [12, Problem XIV, p.12]. The four resulting circles are called a *Descartes quadruple*, and it is impossible to add a fifth. There is a remarkable relationship between their four curvatures (inverse radii):

$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2.$$

René Descartes is first credited with this observation, in correspondence with Princess Elizabeth of Bohemia in 1643 [4, p.49]. Nobel prize winning radiochemist Frederick Soddy published his own rediscovery in *Nature*, shortly before World War II, in the form of a poem [15], which begins:

For pairs of lips to kiss maybe
Involves no trigonometry.
This not so when four circles kiss
Each one the other three . . .

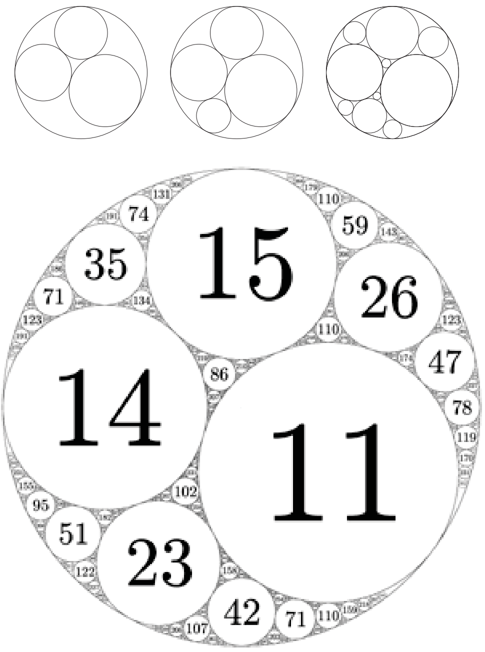

Fig. 1. In the top row, the first few stages of the iterative construction of an Apollonian circle packing. At bottom, an approximation of the finished packing, with curvatures shown. The outer circle has curvature −6, the sign indicating that its interior is "outside."

If we choose three mutually tangent circles of *integer* curvatures a, b and c, then the two solutions of Apollonius correspond to the two solutions to the resulting quadratic equation in d. If one of these is integral, so is the other. Thus we discover that, if we begin with a Descartes quadruple of integer curvatures, we can add a new circle of integer curvature. This quintuple contains several fresh quadruples of integer curvatures, and so follow other new integral circles, in the same manner. Continuing in this way forever, we create an integral Apollonian circle packing, an infinite fractal arrangement of disjoint and tangent circles with integer curvatures, as in Fig. 1.

Related circle arrangements have appeared for centuries, from Japanese *sangaku* temple art [10] to Vi Hart's popular YouTube videos [8]. But it is only in the past decade or two that number theorists have begun to answer the question of which curvatures appear in an integral Apollonian circle packing. It is conjectured [5, 6] that the only obstructions are local: in other words, in a given packing, any sufficiently large integer not ruled out by a specific congruence restriction modulo 24 will appear. This question has become a demonstration piece for the newly emerging theory of thin groups, and sophisticated tools have been brought to bear on its partial solution: namely, that a density one collection of such integers appears [1, 2].

Momentarily putting aside the geometry, the question is a recursive one: given a Descartes quadruple of curvatures a, b, c, d, we obtain a new integer d' satisfying

$$d' + d = 2(a + b + c).$$

The much more classical question of the values represented by a quadratic form is very similar. If

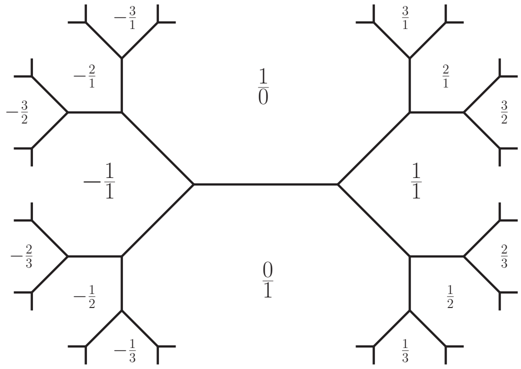

Fig. 2. A portion of Conway's topograph. Each region is labelled by an element of $\widehat{\mathbb{Q}}$. The parallelogram law, with respect to the central wall, reads $f(-1,1)+f(1,1)=2(f(1,0)+f(0,1))$.

f is a quadratic form, then the *parallelogram law* states

$$f(u+v)+f(u-v)=2(f(u)+f(v)).$$

In this way, we can generate the primitive values of an integral binary quadratic form recursively from its values on any triple of primitive vectors $u, v, u+v \in \mathbb{Z}^2$.

This observation led Conway to study these values in visual form on a *topograph* capturing the recursion [3]. Primitive vectors in $\mathbb{Z}^2$ (those with no common factor between their coordinates), considered up to sign, are in bijection with $\widehat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$. The topograph is an infinite tree breaking the plane into regions corresponding to the elements of $\widehat{\mathbb{Q}}$, as in Fig. 2. In this picture, two regions are adjacent (sharing a wall) if and only if the corresponding elements $a/b, c/d \in \widehat{\mathbb{Q}}$ are *unimodular*, i.e. $ad-bc=\pm 1$. The recurrence relation above relates the values of f on the four regions surrounding any one wall (u and v on either side, and $u \pm v$ at either end). In Conway's tree, given two regions that share a boundary edge (are "*tangent*"), there are exactly two ways to choose a third so that all three are mutually "tangent". Three such "mutually tangent" elements of $\widehat{\mathbb{Q}}$ are called a *superbasis*. The values of an integral binary quadratic form on a superbasis determine all the values through the recursion of the parallelogram law.

These two questions — values of a quadratic form and curvatures in a packing — can be unified with a little hyperbolic geometry. Let us consider $\mathcal{H}$, the upper half plane. This is a model of $\mathbb{H}^2$, the hyperbolic plane, in which the geodesics run along Euclidean circles orthogonal to its boundary, $\widehat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$. The hyperbolic isometries are obtained by extending the action of $\mathrm{PSL}_2(\mathbb{Z})$ on the boundary. The action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathcal{H}$ has a fundamental region as delineated by geodesics in Fig. 3. Also shown in that figure is Conway's topograph, now reincarnated as a subset of the walls between images of this region: to find it, choose the walls of finite length, i.e. those not approaching the boundary $\widehat{\mathbb{R}}$. The regions of the topograph now correspond to the cusps in the picture: one for each element of $\widehat{\mathbb{Q}}$. (Each such region has a *horocircle* inscribed in it, a circle tangent to all sides. The collection of such circles (discounting the one at ∞) is famously known as the *Ford circles*.)

Also shown in Fig. 3, in green, is the orbit of the geodesic line from 0 to ∞. The green geodesics are in bijection with the unimodular pairs of $\widehat{\mathbb{Q}}$, by associating to each line its two boundary points. This green structure could be termed a *Farey fractal*, as it illustrates the well-known *Farey subdivision* of $\widehat{\mathbb{R}}$: beginning with the two intervals created by $0=0/1$ and $\infty=1/0$, subdivide each interval $(a/b, c/d)$ at its mediant $(a+c)/(b+d)$. Each green arc is the "top" of a hyperbolic triangle formed by this subdivision. This is called the *upper half plane Farey diagram* in Hatcher's rich treatment [9, Chapter 1].

Now let us move up one dimension. Consider the upper half space $\mathcal{S}$, a model of $\mathbb{H}^3$ sitting above its boundary, $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The geodesic planes are the hemispheres orthogonal to $\widehat{\mathbb{C}}$. The hyperbolic isometries of this model are exactly the extensions of the Möbius transformations on $\widehat{\mathbb{C}}$, which can be expressed as $\mathrm{PSL}_2(\mathbb{C}) \rtimes \langle \mathfrak{c} \rangle$ where $\mathfrak{c}$ is complex conjugation.

The analogue of $\mathrm{PSL}_2(\mathbb{Z})$ of interest to us in this setting is $\mathrm{PSL}_2(\mathcal{O}_K)$ where $\mathcal{O}_K$ is the ring of integers of an imaginary quadratic field. This is a discrete subgroup of hyperbolic isometries, and it has a fundamental domain, which is a volume cut out by several geodesic planes.

There are many analogies to the upper half plane. In $\mathcal{H}$, the set $\widehat{\mathbb{Q}}$ was the orbit of the single cusp of the fundamental region. The number of cusps of the fundamental region of $\mathrm{PSL}_2(\mathcal{O}_K)$ is the class number of $\mathcal{O}_K$. What is the analogue of the Farey fractal? We take the orbit of one geodesic plane. Restricting to the boundary $\widehat{\mathbb{C}}$, this gives an orbit of circles under the collection of Möbius transformations $\mathrm{PSL}_2(\mathcal{O}_K)$. A natural choice is $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$: then we obtain a *Schmidt*

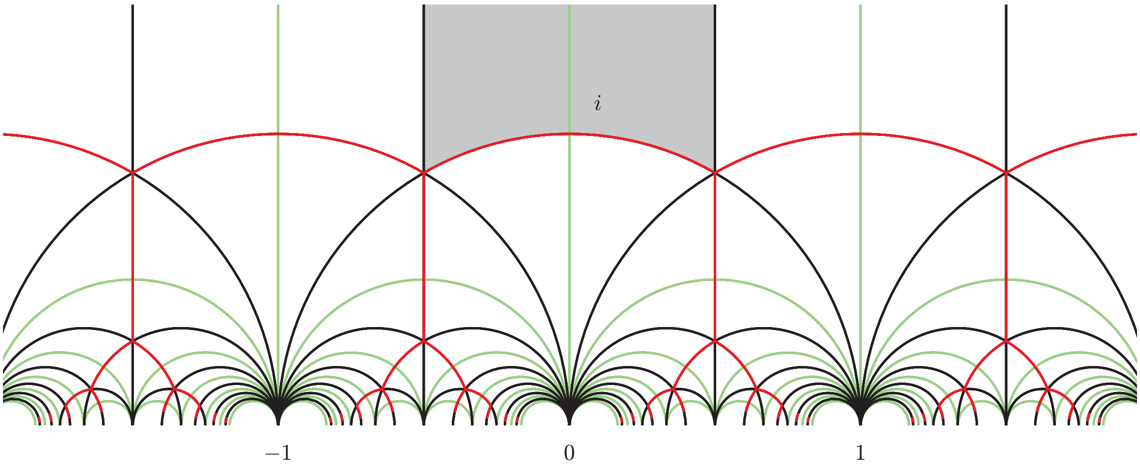

Fig. 3. A portion of the upper half plane $\mathcal{H}$. In shaded grey, the usual fundamental region for the action of $\mathrm{SL}_2(\mathbb{Z})$. In black and red, the boundaries subdividing $\mathcal{H}$ into images of this fundamental region. In red, Conway's topograph. In green, the orbit of the geodesic line from 0 to ∞ (which also breaks up $\mathcal{H}$ into fundamental triangles).

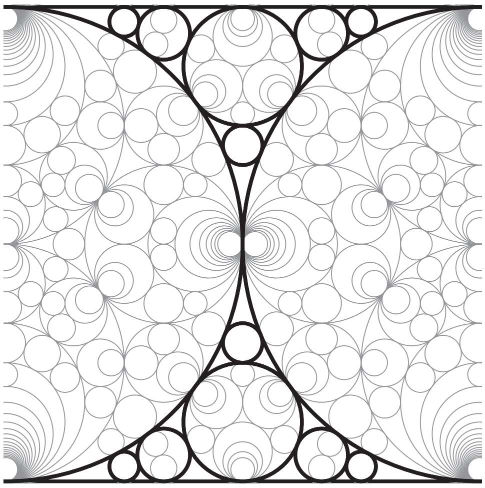

Fig. 4. The Schmidt arrangement of the Gaussian integers. The box between 0 and $1+i$ is shown, including only circles with curvatures at most 20. The Schmidt arrangement is periodic under translation by $\mathbb{Z}[i]$. The Apollonian strip packing (which is bounded by two horizontal lines through 0 and i) is highlighted in black.

arrangement [17, 18], named for Asmus Schmidt's study of complex continued fractions [14]. The Gaussian case, $O_K = \mathbb{Z}[i]$, is shown in Fig. 4.

The circles of the Gaussian Schmidt arrangement are pairwise either disjoint or mutually tangent. They are dense in $\widehat{\mathbb{C}}$ and yet they have a fascinating fractal structure. A circle obtained as the image of $\widehat{\mathbb{R}}$ under a transformation $\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)$ has curvature $i(\beta\overline{\delta} - \overline{\beta}\delta)$, which is always twice a rational integer. Therefore, dilating by a factor of two, we obtain a wild forest of tangent and disjoint circles of integer curvature. Remarkably, it includes every possible integral Apollonian circle packing, up to rigid motions and scaling (see [7, Theorem 6.1] and more generally [18, Theorem 1.3]).

In other words, there is a subgroup of $\mathrm{PSL}_2(O_K)$ which generates any integral Apollonian circle packing. The orbit of this subgroup is shown in Fig. 4. This subgroup is called the *Apollonian group*, and it is a so-called *thin group*, i.e. of infinite index in its Zariski closure.

Similar thin groups appear in other Schmidt arrangements, giving rise to other Apollonian-like packings with integrality properties [18]. The Schmidt arrangements themselves reflect the arithmetic of their respective fields; for example, the Schmidt arrangement of K is connected if and only if O_K is Euclidean [17, Theorem 1.5] (see Fig. 5).

Now let us return to the question of the integers represented by forms and curvatures. First, consider $\mathcal{H}$. The unimodular pair $(a/b, c/d)$ has separation (distance between the elements) $1/bd$. Hence the integral binary quadratic form $f(b,d) = bd$ is a reasonable choice for the meaning of *curvature* of the pair. Given any superbasis — geometrically, a hyperbolic triangle of the Farey fractal — if we know its curvatures, the parallelogram law determines the curvatures of the further subdivisions — the adjacent triangles. The recursive structure of the Farey fractal illustrates exactly this.

Now, consider $\mathcal{S}$. A circle has curvature $i(\beta\overline{\delta} - \overline{\beta}\delta)$. This is a Hermitian form. Knowledge of the form on any Descartes quadruple determines it, recursively, on the entire Apollonian circle packing. The recursive structure

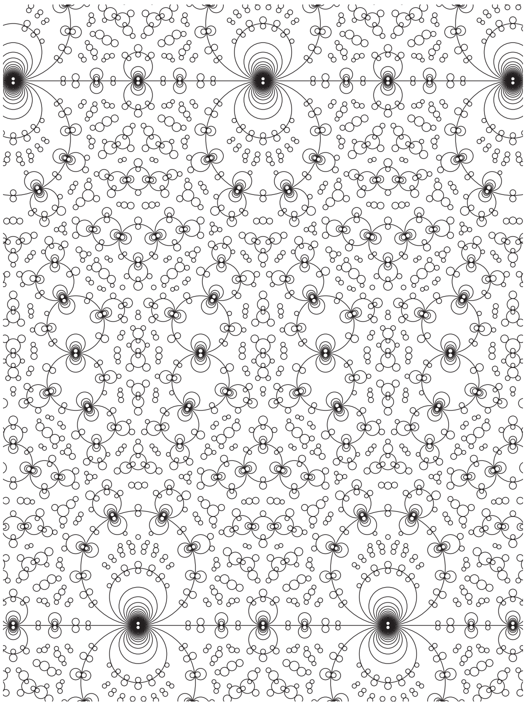

Fig. 5. The Schmidt arrangement of $\mathbb{Q}(\sqrt{-19})$, which is disconnected. Only circles with curvature at most 30 are shown; the two dark accumulations near the bottom are located at 0 and 1.

of the Schmidt arrangement illustrates exactly this.

Therefore we have uncovered a rather pleasing analogy:

$\mathcal{H}$	$\mathcal{S}$
$\widehat{\mathbb{R}}$	$\widehat{\mathbb{C}}$
$\mathbb{H}^2$	$\mathbb{H}^3$
$\mathrm{PSL}_2(\mathbb{Z})$	$\mathrm{PSL}_2(\mathbb{Z}[i])$
unimodular pairs	circles
1/separation	curvature
quadratic form	Hermitian form

For more on this analogy, see [16–18].

However, the difficulty in resolving the local-global conjecture, as compared to describing the values of a quadratic form, resides in the fact that an Apollonian circle packing represents a thin subgroup of $\mathrm{PSL}_2(O_K)$. For more on this fascinating new frontier, see [11].

A note on figures. The figures in this document were produced using Sage Mathematics Software [13].

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