Grothendieck and Algebraic Geometry

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Grothendieck's thesis and subsequent publications in the early 1950's dealt with functional analysis. This was remarkable work, which is attracting new attention today.[a] Still, his most important contributions are in algebraic geometry, a field which occupied him entirely from the late 1950's on, in particular during the whole time he was a professor at the Institut des Hautes Études Scientifiques (IHÉS) (1959–1970).

Algebraic geometry studies objects defined by polynomial equations[b] and interprets them in a geometric language. A major problem faced by algebraic geometers was to define a good framework and develop local to global techniques. In the early 1950's complex analytic geometry showed the way with the use of sheaf theory. Thus, a complex analytic space is a ringed space, with its underlying space and sheaf of holomorphic functions (Oka, H Cartan). Coherent sheaves of modules over this sheaf of rings play an important role.[c] In 1954 Serre transposed this viewpoint to algebraic geometry for varieties defined over an algebraically closed field. He employed the Zariski topology, a topology with few open subsets, whose definition is entirely algebraic (with no topology on the base field), but which is well adapted, for example, to the description of a projective space as a union of affine spaces, and gives rise to a cohomology theory which enabled him, for example, to compare certain algebraic and analytic invariants of complex projective varieties.

Inspired by this, Grothendieck introduced *schemes* as ringed spaces obtained by gluing (for the Zariski topology) spectra of general commutative rings. Furthermore, he described these objects from a *functorial* viewpoint. The language of categories already existed, having appeared in the framework of homological algebra, following the publication of Cartan–Eilenberg's book (*Homological Algebra*, Princeton Univ. Press, 1956). But it was Grothendieck who showed all its wealth and flexibility. Starting with a category C, to each object X of C one can associate a *contravariant functor* on C with values in the category of sets, $h_X : C \to \mathit{Sets}$, sending the object T to $\mathrm{Hom}_C(T, X)$. By a classical lemma of Yoneda, the functor $X \mapsto h_X$ is fully faithful. To preserve the geometric language, Grothendieck called $h_X(T)$ the set of *points* of X with values in T. Thus, an object X is known when we know its points with values in every object T. Grothendieck applied this to algebraic geometry. This was revolutionary as, until then, only field valued points had been considered.

As an example, suppose we have a system of equations

$$f_1(x_1, \ldots, x_n) = \cdots = f_N(x_1, \ldots, x_n) = 0, \qquad (1)$$

where the f_i's are polynomials with coefficients in $\mathbf{Z}$. Let $\mathcal{A}$ be the opposite category of the category of rings, and let F be the (contravariant) functor sending a ring A to the set $F(A)$ of solutions $(x_i), x_i \in A$, of (1). This functor is nothing but the functor h_X for X the object of $\mathcal{A}$ corresponding to the quotient of $\mathbf{Z}[x_1, \ldots, x_n]$ by the ideal generated by the f_i's: the functor F is *represented* by X, an *affine scheme*. Points of X with values in $\mathbf{C}$ are points of a complex algebraic variety — that one can possibly study by analytic methods — while points with values in $\mathbf{Z}$, $\mathbf{Q}$, or in a finite field are solutions of a diophantine problem. Thus the functor F relates arithmetic and geometry.

If the f_i's have coefficients in a ring B instead of $\mathbf{Z}$, the analogous functor F on the category $\mathcal{A}$ opposite to that of B-algebras, sending a B-algebra A to the set $F(A)$ of solutions of (1) with values in A, is similarly represented by the spectrum X of a B-algebra (quotient of $B[x_1, \ldots, x_n]$ by the ideal generated by the f_i's), a scheme *over* the spectrum of B. In this way a *relative* viewpoint appears, for which the language of schemes is perfectly suited. The essential tool is *base change*, a generalisation of the notion of extension of scalars: given a scheme X over S and a base change morphism $S' \to S$, we get a new scheme X' over S', namely, the fiber product of X and S'

[a] See [G Pisier, Grothendieck's Theorem, Past and Present, *Bull. Amer. Math. Soc.* (N.S.) **49**(2) (2012) 237–323].

[b] However, we seldom saw Grothendieck write an explicit equation on the blackboard; he did it only for basic, crucial cases.

[c] Cf. Cartan's famous theorems A and B.

over S. In particular, X defines a family of schemes X_s parameterised by the points s of S. The above functor F then becomes the functor sending a S-scheme T to the set of S-morphisms from T to X. A number of useful properties of X/S (such as smoothness or properness) can nicely be read on the functor h_X. The two above mentioned properties are stable under base change, as is *flatness*, a property that plays a central role in algebraic geometry, as Grothendieck showed. In 1968, thanks to M Artin's approximation theorem, it became possible to characterise functors that are representable by algebraic spaces (objects very close to schemes) by a list of properties of the functor, each of them often being relatively easy to check. But already in 1960, using only the notion of flatness, Grothendieck had constructed, in a very natural way, Hilbert and Picard schemes as representing certain functors, at once superseding — by far — all that had been written on the subject before.

Nilpotent elements[d] in the local rings of schemes appear naturally (for example in fiber products), and they play a key role in questions of infinitesimal deformations. Using them systematically, Grothendieck constructed a very general differential calculus on schemes, encompassing arithmetic and geometry.

In 1949 Weil formulated his conjectures on varieties over finite fields. They suggested that it would be desirable to have at one's disposal a cohomology with discrete coefficients satisfying an analogue of the Lefschetz fixed point formula. In classical algebraic topology, cohomology with discrete coefficients, such as $\mathbf{Z}$, is reached by cutting a complicated object into elementary pieces, such as simplices, and studying how they overlap. In algebraic geometry, the Zariski topology is too coarse to allow such a process. To bypass this obstacle, Grothendieck created a conceptual revolution in topology by presenting new notions of gluing (a general theory of *descent*,[e] conceived already in 1959), giving rise to new spaces: *sites* and *topoi*, defined by what we now call *Grothendieck topologies*. A Grothendieck topology on a category is the datum of a particular class of morphisms and families of morphisms $(U_i \to U)_{i\in I}$, called *covering*, satisfying a small number of properties, similar to those satisfied by open coverings in topological spaces. The conceptual jump is that the arrows $U_i \to U$ are not necessarily inclusions.[f] Grothendieck developed the corresponding notions of sheaf and cohomology. The basic example is the *étale topology*.[g] A seminar run by M Artin at Harvard in the spring of 1962 started its systematic study. Given a scheme X, the category to be considered is that of étale maps $U \to X$, and covering families are families $(U_i \to U)$ such that U is the union of the images of the U_i's. The definition of an étale morphism of schemes is purely algebraic, but one should keep in mind that if X is a complex algebraic variety, a morphism $Y \to X$ is étale if and only if the morphism $Y^{\mathrm{an}} \to X^{\mathrm{an}}$ between the associated analytic spaces is a local isomorphism. A finite Galois extension is another typical example of an étale morphism.

For torsion coefficients, such as $\mathbf{Z}/n\mathbf{Z}$, one obtains a good cohomology theory $H^i(X, \mathbf{Z}/n\mathbf{Z})$, at least for n prime to the residue characteristics of the local rings of X. Taking integers n of the form ℓ^r for a fixed prime number ℓ, and passing to the limit, one obtains cohomologies with values in $\mathbf{Z}_\ell = \varprojlim \mathbf{Z}/\ell^r\mathbf{Z}$, and its fraction field $\mathbf{Q}_\ell$. If X is a complex algebraic variety, one has comparison isomorphisms (due to M Artin) between the étale cohomology groups $H^i(X, \mathbf{Z}/\ell^r\mathbf{Z})$ and the Betti cohomology groups $H^i(X^{\mathrm{an}}, \mathbf{Z}/\ell^r\mathbf{Z})$,[h] thus providing a purely algebraic interpretation of the latter. Now, if X is an algebraic variety over an arbitrary field k (but of characteristic $\neq \ell$) (a k-scheme of finite type in Grothendieck's language), $\bar{k}$ an algebraic closure of k, and $X_{\bar{k}}$ deduced from X by extension of scalars, the groups $H^i(X_{\bar{k}}, \mathbf{Q}_\ell)$ are finite dimensional $\mathbf{Q}_\ell$-vector spaces, and they are equipped with a continuous action of the Galois group $\mathrm{Gal}(\bar{k}/k)$. It is especially through these representations that algebraic geometry interests arithmeticians. When k is a finite field $\mathbf{F}_q$, in which case $\mathrm{Gal}(\bar{k}/k)$ is generated by the Frobenius substitution $a \mapsto a^q$, the Weil conjectures, which are now proven, give a lot of information about these representations. Étale cohomology enabled Grothendieck to prove the first three of these

[d] An element x of a ring is called *nilpotent* if there exists an integer $n \geq 1$ such that $x^n = 0$.

[e] The word "descent" had been introduced by Weil in the case of Galois extensions.

[f] More precisely, monomorphisms, in categorical language.

[g] The choice of the word *étale* is due to Grothendieck.

[h] But not between $H^i(X, \mathbf{Z})$ and $H^i(X^{\mathrm{an}}, \mathbf{Z})$: by passing to the limit one gets an isomorphism between $H^i(X, \mathbf{Z}_\ell)$ and $H^i(X^{\mathrm{an}}, \mathbf{Z}) \otimes \mathbf{Z}_\ell$.

conjectures in 1966.[i] The last and most difficult one (*the Riemann hypothesis for varieties over finite fields*) was established by Deligne in 1973.

When Grothendieck and his collaborators (Artin, Verdier) began to study étale cohomology, the case of curves and constant coefficients was known: the interesting group is H^1, which is essentially controlled by the Jacobian of the curve. It was a different story in higher dimension, already for a surface, and *a priori* it was unclear how to attack, for example, the question of the finiteness of these cohomology groups (for a variety over an algebraically closed field). But Grothendieck showed that an apparently much more difficult problem, namely a relative variant of the question, for a morphism $f : X \to Y$, could be solved simply, by *dévissage* and reduction to the case of a family of curves.[j] This method, which had already made Grothendieck famous with his proof, in 1957, of the *Grothendieck–Riemann–Roch formula* (although the dévissage, in this case, was of a different nature), suggested a new way of thinking, and inspired generations of geometers.

In 1967 Grothendieck defined and studied a more sophisticated, second type of topology, the *crystalline topology*, whose corresponding cohomology theory generalises de Rham cohomology, enabling one to analyse differential properties of varieties over fields of characteristic $p > 0$ or p-adic fields. The foundations were written up by Berthelot in his thesis. Work of Serre, Tate, and Grothendieck on p-divisible groups, and problems concerning their relations with Dieudonné theory and crystalline cohomology launched a whole new line of research, which remains very active today. *Comparison theorems* (solving conjectures made by Fontaine[k]) establish bridges between étale cohomology with values in $\mathbf{Q}_p$ of varieties over p-adic fields (with the Galois action) on the one hand, and their de Rham cohomology (with certain extra structures) on the other hand, thus providing a good understanding of these p-adic representations. However, over global fields, such as number fields, the expected properties of étale cohomology, hence of the associated Galois representations, are still largely conjectural. In this field, the progress made since 1970 owes much to the theory of automorphic forms (*the Langlands programme*), a field that Grothendieck never considered.

In the mid 1960's Grothendieck dreamed of a universal cohomology for algebraic varieties, without particular coefficients, having realisations, by appropriate functors, in the cohomologies mentioned above: the theory of *motives*. He gave a construction, from algebraic varieties and algebraic correspondences between them, relying on a number of conjectures that he called *standard*. Except for one of them,[l] they are still open. Nevertheless, the dream was a fruitful source of inspiration, as can be seen from Deligne's theory of absolute Hodge cycles, and the construction by Voevodsky of a triangulated category of mixed motives. This construction enabled him to prove a conjecture of Bloch–Kato on Milnor K-groups, and paved the way to the proof, by Brown, of the Deligne–Hoffman conjecture on values of multizeta functions.

The above is far from giving a full account of Grothendieck's contributions to algebraic geometry. We did not discuss Riemann–Roch and K-theory groups, stacks and gerbes,[m] group schemes (SGA 3), derived categories and the formalism of six operations,[n] the *tannakian* viewpoint, unifying Galois groups and Poincaré groups, or *anabelian geometry*, which he developed in the late 1970's.

All major advances in arithmetic geometry during the past forty years (proof of the Riemann hypothesis over finite fields (Deligne), of the Mordell conjecture (Faltings), of the Shimura–Taniyama–Weil conjecture (Taylor–Wiles), works of Drinfeld, L Lafforgue, Ngô) rely on the foundations constructed by Grothendieck in the 1960's. He was a visionary and a builder. He thought that mathematics, properly understood, should arise from "natural" constructions. He gave many examples where obstacles disappeared, as if

[i]The first one (*rationality of the zeta function*) had already been proved by Dwork in 1960, by methods of p-adic analysis.

[j]At least for the similar problem concerning *cohomology with proper supports*: the case of cohomology with arbitrary supports was treated only later by Deligne using other dévissages.

[k]The so-called C_{cris}, C_{st}, and C_{dR} conjectures, first proved in full generality by Tsuji in 1997, and to which many authors contributed.

[l]The *hard Lefschetz* conjecture, proved by Deligne in 1974.

[m]These objects had been introduced by Grothendieck to provide an adequate framework for non abelian cohomology, developed by J Giraud (*Cohomologie non abélienne*, Die Grundlehren der mathematischen Wissenschaften **179**, Springer-Verlag, 1971). Endowed with suitable algebraic structures (Deligne–Mumford, Artin), stacks have become efficient tools in a lot of problems in geometry and representation theory.

[n]Currently used today in the theory of linear partial differential equations.

Alexander Grothendieck at the IHÉS in 1960's during his famous Seminar of Algebraic Geometry. [Photo courtesy IHÉS]

by magic, because of his introduction of the right concept at the right place. If during the last decades of his life he chose to live in extreme isolation, we must remember that, on the contrary, between 1957 and 1970, he devoted enormous energy to explaining and popularising, quite successfully, his point of view.

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Grothendieck's three major works in algebraic geometry are:

ÉGA: *Éléments de géométrie algébrique*, rédigés avec la collaboration de J. Dieudonné, Pub. Math. IHÉS 4, 8, 11, 17, 20, 24, 28 et 32.

FGA: *Fondements de la géométrie algébrique*, Extraits du Séminaire Bourbaki, 1957–1962, Paris, Secrétariat mathématique, 1962.

SGA: *Séminaire de Géométrie Algébrique du Bois-Marie*, SGA 1, 3, 4, 5, 6, 7, Lecture Notes in Math. 151, 152, 153, 224, 225, 269, 270, 288, 305, 340, 589, Springer-Verlag; SGA 2, North Holland, 1968.

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