Irrationals in Ancient India

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We discuss the question: In which century were irrational numbers known to Indians? This study becomes a bit complicated when we distinguish the knowledge of irrationality of a number from the knowledge of an irrational number. The irrational numbers like π and √2 were known about three millennia ago, closer and closer rational approximations to them were also attempted long ago, but it is doubted whether the concept of irrationality was known to those who made these attempts.

In this article we discuss this question with special reference to three passages, one from Baudhayana sulvasutra, another from Aryabhateeyam, and the third from Tantrasangraham. We start with the last of them, where the knowledge of irrationality in the fifteenth century is very clear.

> कुतः पुनर्वास्तवीं संख्यां उत्सृज्य आसन्नैवेहोक्ता?
> तस्या वक्तुं अशक्यत्वात् । कुतः? येन मानेन
> मीयमानः व्यासः निरवयवः स्यात् , तेनैव
> मीयमानः परिधिः पुनः सावयवः स्यात् ।
> येन च मीयमानः परिधिः निरवयवः तेनैव
> मीयमानो व्यासोपि सावयवः । इत्येकेनैव मानेन
> मीयमानयोः उभयोः क्वापि न निरवयवत्वं
> स्यात् । महान्तमध्वानं गत्वापि अल्पावयवत्वमेव
> लभ्यम् । निरवयवत्वं तु क्वापि न लभ्यम् । इति
> भावः ।

"Why then is it that discarding exact value, only the approximate one has been mentioned here? This is the answer: because it (the exact value) cannot be mentioned. If the diameter, measured with respect to a particular unit of measurement, is commensurable, then w.r.t. the same unit, the circumference cannot be exactly measured; and if w.r.t. any unit the circumference is commensurable, the diameter cannot be measured. Thus there will never be commensurability for both w.r.t. the same unit of measurement. Even after going a long way, the degree of commensurability can be made very small, but absolute commensurability can never be attained".

– Translation by Prof C N Srinivasiengar in 1967

When was this book written? We have to look into some other books by the same author. In one of his works titled *Siddhanta-saara* and also in his own commentary on *Siddhanta-darpana*, Nilakantha Somayaji, the author of Tantrasangraham, has stated that he was born on Kali-day 1,660,181 which works out to June 14, 1444 CE. Undoubtedly, Indians were aware of irrational numbers at this time. Were they knowing it still earlier?

This passage, quoted above in Sanskrit, itself is a commentary to a sloka in Aryabhateeyam, which runs as follows:

> चतुरधिकं शतमष्टगुणम्
> द्वाषष्टिस्तथा सहस्राणाम् ।
> अयुतद्वयविष्कम्भ
> स्यासन्नो वृत्त परिणाहः ॥

This is translated as follows:

"Add four to 100, multiply by eight and then add 62,000. By this rule the circumference of a circle of diameter 20,000 can be approached".

This means that the commentator (Nilakantha) contends that Aryabhata (499 AD) was aware of the notion of irrationality, and that his word "aasanna" (approximate) discloses this. But just because the number 62432/20000 was stated as an approximation to π, can we conclude that the author knew that π was not rational? Is it not possible that simpler rational approximations are being provided to more complicated rational numbers? This objection has been voiced out in the following review:

"It is true that Aryabhata used the word asanna ('approximately') for his excellent value π = 62832/20000 but there is no evidence to show that π was regarded as irrational by Aryabhata himself."

This leads us to examine what other historians think about the question whether Aryabhata had the knowledge of irrational numbers. We quote two passages in this connection.

"It is speculated that Aryabhata used the word āsanna (approaching), to mean that not only is this an approximation but that the value is incommensurable (or irrational). If this is correct, it is quite a sophisticated insight".

– Wikipedia

"Further to deriving this highly accurate value for π, Aryabhata also appeared to be aware that it was an 'irrational' number and that his value was an approximation, which shows incredible insight. Thus even accepting that Ptolemy discovered the 4 decimal place value, there is no evidence that he was aware of the concept of irrationality, which is extremely important."

– Mac Tutor, History of mathematics website

It is thus clear that historians are divided on this issue. But one point deserves a mention. There has been a stream of commentaries on Aryabhatiya, and some of them have maintained that Aryabhata was aware of the incommensurability of π with 1.

"The Aryabhatiya was an extremely influential work as is exhibited by the fact that most notable Indian mathematicians after Aryabhata wrote commentaries on it. At least twelve notable commentaries were written for the Aryabhatiya ranging from the time he was still alive (c. 525) through 1900 ('Aryabhata I' 150-2). The commentators include Bhaskara and Brahmagupta among other notables".

– William Gongol

This number π has attracted the attention of so many Indian mathematicians of several centuries, who have come out with so many different rational approximations.

• Yajnavalkya (estimated: 800 BC)	• 339/108
• Bodhayana (estimated: 800 BC)	• 4(1–1/8+(1/8.29)–(1/8.29.6))
• Manava (estimated: 500 BC)	• 3+(1/5)
• Aryabhata (499 AD)	• 62832/20000
• Brahmagupta (628 AD)	• Square root of 10
• Bhaskara (1150 AD)	• 22/7 or 3927/1250
• Jyeshtadeva (1500–1600 AD)	• 31415926536/10000000000
• Putumana Somayaji (1660–1740 AD)	• $\sqrt{12}$(1–1/3.3+1/5.3^2–…)
• Sankaravarman (1800–1838 AD)	• 3.14159265358979324
• Bharati Krishna Teertha (1884–1960)	• 3.1415926535897932384626433832 79

Of these ten, all but the first three belong to the post-Christian era. For the first three, the estimates vary too much, ranging from 3000 BC to 500 BC. All their ten books, except the last one, have been composed in Sanskrit language. In the last also, this result has been given in the form of a Sanskrit verse. The list reveals the fact that the approximations are better in later periods, as can be expected in any civilisation.

Now let us pay some attention to the following passage from Baudhayana Sulvasutra, (whose period is uncertain, but estimated as about 800 BC by many historians) giving a rational approximation for the square root of two.

प्रमाणं तृतीयेन वर्धयेत् तच्चतुर्थेन
आत्मचतुस्त्रिंशोनेन सविशेषः

Here is its translation as given in [3]: "The measure is to be increased by the third and this (third) again by its own fourth less the thirty fourth part (of that fourth); this is (the value of) the diagonal of a square (whose side is the measure)." We note that here the last word "savishesha" has been left untranslated. This amounts to say that $\sqrt{2}$ is equal to

1+1/3 +(1/3)(1/4) – (1/3)(1/4)(1/34) +etc.

The following observations are now in order. (1) The above passage clearly says that there are more terms in this series. The commentators have also interpreted it so. Some of them have attempted to suggest what the fifth or sixth term could be. (2) There have been different approaches to guess the fifth term, but many of them arrive at the same answer, namely –(1/3)(1/4)(1/34)(1/1154); all these approaches are indicated in a natural manner, by a close examination of the first four terms in the Sulvasutra. (3) It is seen that the partial sums in this series, (from the third onwards) share a common property; the square of the numerator differs from twice the square of the denominator by exactly 1. (4) The geometric method in [1] and the number theoretic method in [3] indicate that the same is happening after any number of terms.

Any serious learner will surely suspect that Bodhayana has deliberately given his initial four terms by a definite rule and that this rule itself demonstrates the irrationality of $\sqrt{2}$.

Summary: Aryabhata's passage alone cannot confirm his knowledge of irrationality of π, but some of his commentators do affirm that Aryabhata was aware of this notion. Baudhayana's passage is still more vulnerable, but an examination of the pattern in its series shows that he might have been aware of the irrationality of $\sqrt{2}$. Irrationality was known in India, in the fifteenth century, beyond a ray of doubt; in the fifth century itself, as per some Sanskrit books; possibly in 800 BC itself, as this cannot be completely ruled out. It is better to leave this discussion at this stage without venturing to answer our question conclusively.

References

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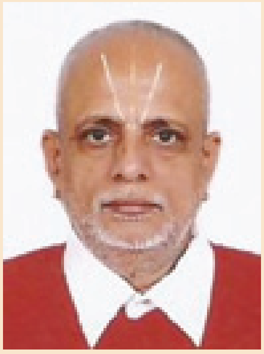
[2] S. N. Sen and A. K. Bag, *The Sulbasutras* (INSA, Delhi, 1983).
[3] V. Kannan and S. Gopal, A hidden algorithm in the Sulvasutras, in *Rashtriya Samskrita Vidyapeeta Publication series* 265 (2010) 17–26.



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