Computational Imaging and Partial Differential Equations

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1. Introduction

Digital images are generated through quantisation and data extraction from analog images for the real world by image capture devices such as a digital camera. The basic idea is to introduce a regular grid to the analog image and to assign a natural number 0 ~ 255 for each square of the grid, which is called a pixel. A numerical value in each pixel represents the brightness or the average of gray-level of the image in the square. Sometimes, it uses multiple channels, for example, in the case of a colour image, three channels red, green, and blue are used. Another important aspect in the digital image is the resolution, which indicates the compactness of the grid. The higher resolution the digital image has, the closer it is to the real world.

Since the image has various structures such as resolution, texture, and appearance like objects in the real world, it is very complex. Therefore, it is very difficult that we look for an approach to handle the several structures at the same time in processing the digital image. Many methods have been developed thus far, however, it is not easy to determine which one is more natural than others. Image processing has a long history and there have been empirical methods and also theoretical ones based on basic probability, statistics, filter theory, and the spectral analysis. These methods, for example, the methods according to simple experiences such as histogram equalisation give plausible results at first, but serious constraints appear since there is no analysis for why the method works, when it works or when it does not work. In recent years, extremely sophisticated methods using wavelets, advanced probability model, or partial differential equations have been developed in order to overcome these difficulties. Since the 1990s, partial differential equation approaches, in particular, have been developed intensively.

Partial differential equation (PDE), which is one of the most important areas of mathematics, is closely related to the world where physical laws govern. Scientists dealing with natural phenomena know basic equations such as heat equation, wave equation, Euler equation, Poisson equation, and Laplace equation. Origin of PDEs is physics, but, recently PDEs are frequently applied to image processing, life sciences, even to the financial market analysis. The main reason for using PDEs in the image processing is that it is possible to analyse the image in continuous spaces. The thought in a continuous space provides intuition to facilitate the understanding of the physical phenomena and to develop a new method. In addition, for PDEs of the continuum mechanics, a lot of theory such as the uniqueness and the existence of the solution are already well developed.

Since PDEs hold for a continuum, to use those for image processing, we assume that the image is defined on a rectangular region instead of a grid, induce the necessary PDEs, and find a numerical solution through the discretisation process again. It is another advantage of the PDE approach, therefore, that we are able to obtain reliable results by using the various conventional, fast and accurate numerical methods. Of course, when considering the numerical solution of PDEs for image processing, it is necessary to create PDEs with appropriate structure since we have to consider the derivatives of non-smooth images. However, even if the image is not smooth enough to be differentiable in the classical sense, the theory for the viscosity solution of hyperbolic PDEs provides us with the theoretical basis for the rigorous usage of PDEs.

A PDE in the image processing is mainly generated from the variational problem

$$\min_I \mathcal{L}(I),$$

where $\mathcal{L}$ is an energy that is calculated on the

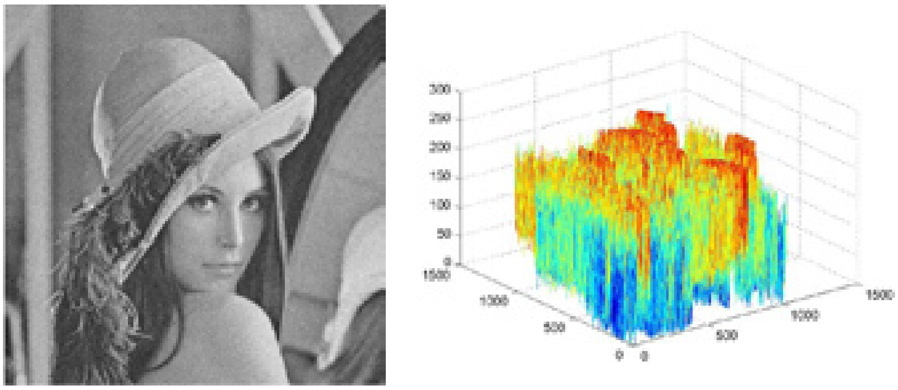

Fig. 1. Lena image with noise and its graph in three dimensions.

image I. If we let $\mathcal{J}(I)$ the variational derivative of $\mathcal{L}$, the minimum can be calculated as the steady-state solution of

$$\frac{\partial I}{\partial t} = -\mathcal{J}(I)\,, \tag{1.1}$$

where t is the artificial time variable. It is, thereby, possible to express curves, surfaces, and images by PDEs, and to obtain the desired results by the solution of such PDEs. In the case of a colour image, we need a system of PDEs.

2. Basic of Computational Imaging: Denoising

The problem that we encounter most often in the image processing is denoising. Figure 1 is the Lena image that is most frequently used as an example in the field of image processing; it shows noisy one and its three dimensional graph.

The easiest way of removing noise is to select the average of the image locally in each pixel. In terms of computational imaging, this can be expressed as the heat equation

$$u_t - \Delta u = 0\,,$$
$$u(0,x) = u_0(x)\ \text{(noisy image)}\,.$$

According to the basic theory of PDEs, the solution of this equation is expressed using the heat kernel as follows:

$$u(t,x) = \int_{\mathbb{R}^d} K(t, x-y)u_0(y)\,dy\,,$$
$$K(t,x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{\|x\|^2}{4t}}\,.$$

In statistics, it is referred to the Wiener filter and discontinuous functions cannot be the solution.

Another way to remove the noise is to solve the following least squares problem with a constraint when the noise level σ is known:

$$\min_{\|u-u_0\|=\sigma} \|\nabla u\|^2\,. \tag{2.1}$$

In other words, while maintaining the distance σ to the noisy image u_0, we find as smooth u as possible. This problem, by introducing Lagrange multipliers, can be replaced by the min-max problem

$$\min_u \max_\lambda \left\{ \mathcal{L}(u,\lambda) = \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\lambda\left(\|u-u_0\|^2 - \sigma^2\right)\right\}.$$

By applying a variational technique here, we obtain the following PDE with a constraint:

$$-\Delta u + \lambda(u-u_0) = 0\,,$$
$$\|u-u_0\| = \sigma\,.$$

Even though both of these methods remove the noise in the image, they have a side effect that sharp edges are lost. As a denoising method to keep edges, we consider a modified minimisation problem

$$\min_{\|u-u_0\|=\sigma} \int_\Omega |\nabla u|\,,$$

where Ω is an image domain. The difference with (2.1) is that it minimises the total variation that uses L^1-norm instead of L^2-norm of ∇u. Thus, it is possible to obtain a discontinuous solution. After we apply the variational technique to the min-max problem with the Lagrange multiplier as described above, ignoring the noise level σ for the convenience of calculation, we get the ROF (Rudin–Osher–Fatemi) model [9]

$$-\nabla\cdot\left(\frac{\nabla u}{|\nabla u|}\right) + \lambda(u-u_0) = 0 \quad \text{in } \Omega\,,$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega\,.$$

The solution of this problem can be found using the conventional gradient descent method as in

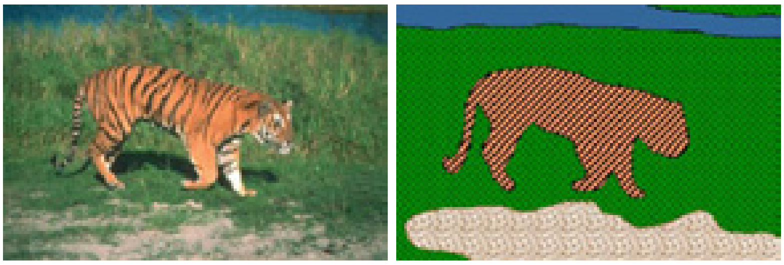

Fig. 2. Region-based segmentation.

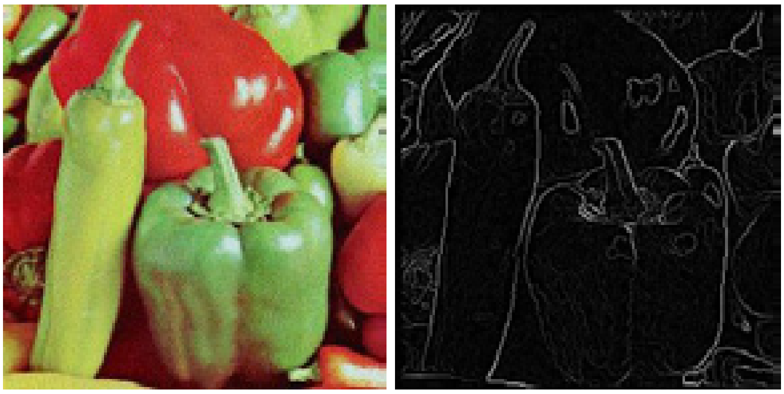

Fig. 3. Edge-based segmentation.

(1.1):

$$u_t = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) - \lambda(u - u_0) \quad \text{in } \Omega\,,$$

$$u(0, x) = u_0(x) \quad \text{in } \Omega\,,$$

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega\,.$$

In denoising, an alternative to formulating a variational problem and inducing the Euler–Lagrange equation is adjusting coefficients of a certain PDE directly. Perona–Malik model [8] is a typical example.

3. Basic of Computational Imaging: Segmentation

In addition to the denoising problem, one of the most important problems in image processing is the segmentation problem. By segmentation, we want to find the boundary of objects in an image. Methods for segmentation are classified into two categories, region-based and edge-based. Figures 2 and 3 show the difference between these two categories.

The simplest method of image segmentation is the thresholding to separate an image by an appropriate value in the histogram of the image values. However, because boundary curves are not connected usually, there is a disadvantage to connect the boundary curves through the post-processing. On the other hand, when using the geometric snake which is one of PDE based image segmentation methods, it no longer requires such a post-treatment, thus, many PDEs that implement geometric snake have been proposed. Here, the snake means a closed curve that is not twisted.

The basic idea of how to find the snake is based on the fact that the boundary of an object in an image is determined by the rapid change of the image values. In other words, the slope would be infinite on the boundary. Therefore, we can match a deformable curve to the boundary of the object through the minimisation process of the energy involving the slope of the image. This method is a typical edge-based method of image segmentation.

Kass–Witkin–Terzopoulos model [5] is the first method in this direction. In this model, the energy

$$J_1(C) = \int_a^b |C'(q)|^2\, dq + \beta \int_a^b |C''(q)|^2\, dq + \lambda \int_a^b g^2(|\nabla I(C(q))|)\, dq$$

over a piecewise regular curve C is minimised. In the third term, g is a monotonically decreasing

Fig. 4. Inpainting: Restoration of a damaged image.

function that becomes zero at infinity, and the curve C is drawn to the boundary of the object by this term. However, this model has an essential disadvantage that the energy J_1 varies with the parameterisation q, therefore, Caselles, Kimmel, and Sapiro suggested the geodesic active contour (GAC) model [2]

$$\min_C \left\{ J_2(C) = \int_a^b g(|\nabla I(C(q))|)|c'(q)|\,dq \right\}.$$

If we select $\beta = 0$ from the energy J_1, minimising $J_1(C)$ and $J_2(C)$ are equivalent. The energy J_2 is invariant under the change of parameterisation and it is an Euclidean length with weight. This energy was also proposed by Kichenassamy *et al.* [4] at the same time. To find a minimiser of J_2, we evolve a curve $C(t,q)$ along the decreasing gradient flow

$$\frac{\partial C}{\partial t} = (\kappa g - \langle \nabla g, N \rangle)N$$

with an initial curve $C_0(q)$, where N is the outward normal vector to $C(t,q)$ and κ is the curvature. By the property of g, when the boundary of an object is found, the evolution of the curve $\{C(t,q)\}_{t\geq 0}$ is stopped. However, it has also a disadvantage because this model is due to the parameterisation. For example, during the evolution of the curve, topological change that may occur in the case of multiple targets is not permitted. To overcome such a problem Sethian and Osher proposed the level set method [7].

There are also models for denoising and segmentation both. As a typical example, there is Mumford–Shah model [6]

$$\min_{\Gamma,u} \left\{ \lambda \int_{\Omega\setminus\Gamma} \nabla u|^2 + \mu|\Gamma| + \int_\Omega |u - u_0|^2 \right\}.$$

It was developed to the model in [1] by coupling with the GAC model, and also was grown to the active contour without edges model [3] considering both the edge and region.

As a typical example of the region-based approach, there is the region-aided geometric snake (RAGS) proposed by Mirmehdi and Xie [10]. This method, using a region map, expands a region from a small specified region.

4. Various Topics of Computational Imaging

In the computational imaging, in addition to denoising and segmentation, there are a wide variety of topics such as inpainting, deblurring, super resolution, texture separation, image sequence analysis, classification, optical flow, image registration, three-dimensional expression and so on. Figure 4 is an example of inpainting to restore the damaged image and Fig. 5 is an example to put a 3-D effect to the video scenes of the movie Avatar.

Image segmentation, in particular, is an essential tool in editing videos. For example, in weather forecast, to predict moving of the cloud, it is necessary to segment the image. Further, by having more information about the background in an image using segmentation, it is possible to reduce the whole process of more complex work such as compression. Further, it can be used in recognition on the web, 3-D restoration, robotics, and analysis of the lines of the palm.

In the medical field, a number of imaging equipments, such as ultrasound, X-rays, CT, and MRI, are used. For the improvement of image quality, enhancement of any characteristic, and mixing of various image information, images

Fig. 5. Image that contains three-dimensional depth.

generated in these devices undergo the image processing. Denoising, segmentation, and image registration, in particular, play a very important role in the interpretation of information from the medical imaging devices.

As the world is developed, the role of images is increasing further. We express many things in the world by images, exchange data in the form of images, and in addition, obtain the necessary information by analysing the image data. This trend will continue to expand, therefore, the role of the computational imaging will also do. Further, it is expected that fast methods from the large scale high performance computing will be introduced in the field of computational imaging as the size of an image becomes larger and the amount of image data becomes bigger than ever.

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