# Achievements of Kazuya Kato

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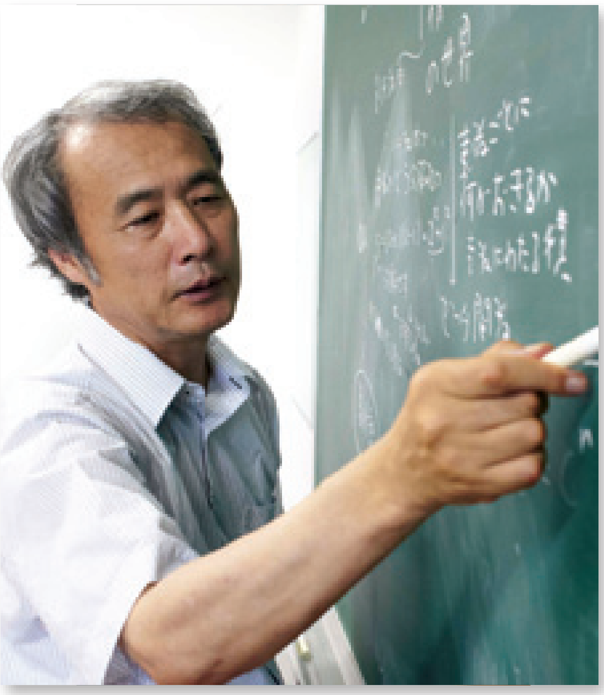

Kazuya Kato

The achievements of Kazuya Kato range over several branches of mathematics. It has inspired and will continue to inspire the whole mathematical community over many years. Here, efforts are made by an inexperienced mathematician like me to explain what he has done, and by this, only a very tiny part of the vast expanse of the mathematical picture drawn on Kato's mind would be revealed. I would like to ask the readers to accept my apology for my incompetence. Before describing Kato's mathematical achievements, I would like to write on his biography and personal character.

Kato was born on January 17, 1952 in the prefecture of Wakayama of Japan and grew up in the prefecture of Ehime. He attended college at the University of Tokyo, from which he also obtained his master's degree in 1975, and his PhD in 1980. He was a professor at Tokyo University, Tokyo Institute of Technology and Kyoto University. He joined the faculty of the University of Chicago in 2009. Kato received numerous prizes among which is the Imperial Prize of the Japan Academy for "Research on Arithmetic Geometry" that he received in 2005.

Kato loves nature. I was fortunate to be Kato's first graduate student at University of Tokyo. Seminars were sometimes held in mountains outside of the city, called the field seminar. Looking at the scene of mountains, Kato said, "When studying mathematics, one's heart resembles the child's heart wondering what is beyond that mountain". It happened more than 30 years ago but I still remember the clear vision when I understood that a certain cohomology group contains a vector space of infinite dimension over a finite field, the vision of a chain of an infinite number of finite fields hanging over the beautiful valley in front of my eyes.

The warmth of Kato's heart appears in his unstinting assistance to his students and others. Kato does not hesitate to share his ideas which gush like a spring with others. It is hard to imagine how many people have done researches based on Kato's ideas (I am also one of them). The number may exceed that of his own papers.

Kato sometime uses metaphors from folktales and compose poems in effort of conveying the essence of intricate mathematical concepts. He says, "Mathematics and art take root in the same origin". Kato's words "a starlit sky addresses a mathematical theorem" expresses his belief that mathematics helps us to touch the beauty of nature and the mystery of the universe. The special volume of Documenta Mathematica published in honour of his 50th birthday contains Kato's song on Prime Numbers together with research papers written by leading number theorists.

Now I start explaining Kato's works. They are very original and so influential as to create new mainstream areas in mathematics. The ideas in those works have been leading arithmetic and algebraic geometry for many years. It is impossible to explain all of those in this article and I pick up the following works.

- Higher dimensional class field theory
- $p$-adic Hodge theory

- Tamagawa number conjecture on special values of zeta functions
- Iwasawa theory of modular forms and the Birch–Swinnerton–Dyer conjecture

I would like to ask the readers to be aware that there are several other outstanding works of Kato not explained in this article. For example:

- Log geometry and log Hodge theory
- Log abelian varieties and their moduli space
- Ramification theory for algebraic varieties
- Non-commutative Iwasawa theory

## 1. Higher Dimensional Class Field Theory

Class field theory describes the abelian extensions of a field in terms of information inherent to the given field. Classical class field theory for an algebraic number field originates from work of Gauss and was established by Artin and Takagi. It is a fundamental pillar of number theory. Kato's first work is a generalisation of class field theory to higher dimensions. In his master thesis, he introduced the notion of a higher dimensional local field and established the class field theory for it. The technical heart of the work is a heavy computation of the Galois cohomology of higher dimensional local fields. I have heard that he consumed one notebook everyday for the computation. It became the foundation of the later work of Kato and Spencer Bloch on $p$-adic Hodge theory. After higher dimensional local class field theory was established, Kato worked on higher dimensional global class field theory, which is class field theory for finitely generated fields. Eventually this was established as geometric class field theory which controls the abelian coverings of an arithmetic scheme (which means a scheme of finite type over an integer ring) by an idele class group defined using algebraic $K$-theory (I had the good fortune of being a coworker of this work).

## 2. $p$-adic Hodge Theory

After higher dimensional class field theory, Kato started studying $p$-adic étale cohomology of algebraic varieties. The techniques developed in his work on higher dimensional local class field theory played an essential role. Étale cohomology theory, originally invented by Grothendieck, is the key tool in Deligne's proof of the Weil conjecture. An important feature is the natural action of the Galois group of the base field, which makes étale cohomology a very useful tool to study arithmetic questions. The theory of Grothendieck and Deligne provides a good understanding of $\ell$-adic étale cohomology of an algebraic variety over a finite field of characteristic $p$ or a $p$-adic field as long as $\ell \neq p$. However, in order to understand $p$-adic étale cohomology of an algebraic variety over a $p$-adic field, a different new theory was needed to be built, which is now called $p$-adic Hodge theory. A seminal work had been done by John Tate. He studied $p$-adic étale cohomology of an abelian variety over a $p$-adic field and proposed a conjecture (now called the Hodge–Tate conjecture) on the structure of $p$-adic étale cohomology of a general algebraic variety over a $p$-adic field.

In a joint work with Spencer Bloch, Kato made significant contributions to the Hodge–Tate conjecture. The work not only contributes to the conjecture, but has become a milestone in the development of the $p$-adic Hodge theory, which is now one of the most important theories in arithmetic geometry. An anecdote which tells how strong the impact of the result of Bloch and Kato was, is that Deligne fell on the floor in surprise, during a lecture at IHES delivered by Kato on this result.

## 3. Tamagawa Number Conjecture on Special Values of Zeta Functions

Next Kato's research goes to study of special values of zeta functions. The problem originates from Euler who succeeded in expressing the values at positive even integers of the Riemann zeta function using the Bernoulli numbers. The Riemann zeta function has been generalised in various forms and the problem has developed in connection with it. In the second joint work with Bloch, Kato succeeded in formulating a general conjecture (called Tamagawa number conjecture) which describes special values of the zeta functions of an algebraic variety over a number field. Moreover Kato generalised the conjecture further in a form where the Galois action is considered. The conjecture called the equivariant Tamagawa number conjecture is also viewed as a generalisation of the Iwasawa main conjecture and now has become one of the central problems in number theory. In these works, $p$-adic Hodge theory has played an essential role. Remembering that Kato has played a central role also in the development of $p$-adic Hodge theory, it must be said that Kato's research goes on the royal road of mathematics.

## 4. Iwasawa Theory of Modular Forms and the Birch–Swinnerton–Dyer Conjecture

Kato's research takes other more surprising developments. He made a breakthrough for the Iwasawa main conjecture of modular forms. The conjecture was formulated by Barry Mazur in the 70's and claims that two ideals of quite different origins (Selmer groups and $p$-adic zeta functions) are equal. Kato proved that one is contained in another. This is a deep astonishing result and was the best since Mazur proposed the conjecture. Kato was motivated by a work of Beilinson who constructed a distinguished element in an algebraic $K$-group of a modular curve and discovered that it is related to the value at $s = 2$ of the zeta function of a modular form through the regulator map. Kato refined Beilinson's element by an innovative idea to a system of elements called Euler system, and showed that it is related to the value at $s = 1$ of the $p$-adic zeta function of the modular form through the $p$-adic regulator map. By using the Euler system, Kato succeeded also in giving another proof of a theorem of Kolyvagin on the Birch–Swinnerton–Dyer conjecture on the zeta function of an elliptic curve, and also generalising it. Its proof is very intricate and uses the $p$-adic Hodge theory and the explicit formula of the reciprocity map for higher dimensional local fields. Although Kato was motivated by Beilinson's result which was related to the value at $s = 2$ of the zeta function of a modular form, he obtained results related to the value at $s = 1$. Kato expresses it as the $p$-adic warp navigation.

Kato thinks that his Euler system may be thought as an incarnation of a zeta function. To explain this, he sometime quotes a Japanese folktale "Grateful Crane", the story of a crane changing her figure into a princess to return kindness of an old couple. I feel that this way of expression of mathematics fits very well with the beauty of the work of Kato.

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