Rota's Conjecture

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In 1970, Gian-Carlo Rota posed a beautiful conjecture that provides a combinatorial characterisation of linear dependence in vector spaces over any given finite field. Recently Jim Geelen, Bert Gerards and I completed a 15-year project that culminated in a proof of Rota's conjecture. What follows is an attempt to give some insight, in language suited to a general mathematical audience, to Rota's conjecture itself together with a short discussion of the techniques that lead to its proof.

Collinearity, coplanarity and their higher dimensional analogues can be thought of as describing the *combinatorial* properties of a set of points in space. These properties can be captured, through the use of homogeneous coordinates, by linear independence. Thus, given a finite set E of vectors, the combinatorial properties of the vectors in E are captured by knowing those subsets of E that are linearly independent. The properties that we capture in that way are, of course, the properties of interest in classical projective geometry.

In 1935 Hassler Whitney attempted to capture this axiomatically. We are given a set E and a collection of subsets $\mathcal{I}$ of E that we call *independent*. We will say that the pair $(E, \mathcal{I})$ is *representable over a field* $\mathbb{F}$ or $\mathbb{F}$*-representable* if we can find a bijection from E to a collection of vectors in a vector space V over $\mathbb{F}$ such that the independent sets are precisely the sets that are mapped to linearly independent subsets of V. The pair $(E, \mathcal{I})$ is *representable* if there exists a field over which it is representable. Whitney observed that the following three conditions are necessary for $(E, \mathcal{I})$ to be representable.

1. The empty set is independent.
2. Subsets of independent sets are independent.
3. If I and J are independent and $|J| > |I|$, then there is an element $x \in J - I$ such that $I \cup \{x\}$ is independent.

If the above properties are satisfied, then we say that $(E, \mathcal{I})$ is a *matroid*. While the three conditions above are necessary for representability, they are by no means sufficient. It is easy to construct matroids that are not representable and Whitney posed the problem of characterising the matroids representable over a given field.

For infinite fields the evidence seems to suggest that there is no nice answer to Whitney's problem. For this reason, most attention in matroid theory has focused on finite fields. Whitney himself characterised the matroids representable over the two-element field, although a much nicer characterisation was found by Bill Tutte in 1958.

Consider an example. Let E be a 4-element set and let $\mathcal{I}$ be all subsets of E with at most two elements. Geometrically the matroid $(E, \mathcal{I})$ represents the 4-point line. This matroid is traditionally denoted $U_{2,4}$. The matroid $U_{2,4}$ is representable over any field except $GF(2)$. This is because lines in binary space have only three points.

With typical prescience, Bill Tutte proved that $U_{2,4}$ played a fundamental role in the class of binary matroids. To understand that role we need to introduce our concept of substructure and discuss "minors". If e is an element of the matroid M we can *delete* e to obtain a matroid whose independent sets are the independent subsets of M contained in $E - \{e\}$. Eliding a nuance we can *contract* e to obtain a matroid whose independent sets are the subsets A of $E - \{e\}$ having the property that $A \cup \{e\}$ is independent in M. Geometrically contraction corresponds to projecting the points of $E - \{e\}$ from the point e onto a hyperplane. Any matroid obtained from M by a sequence of deletions and contractions is a *minor* of M. Space forbids me to put forward the reasons here; but minors really are the natural notion of substructure for matroids.

A class $\mathcal{M}$ of matroids is *minor closed* if every minor of a member of $\mathcal{M}$ is also in $\mathcal{M}$. The matroid M is an *excluded minor* for the minor-closed class $\mathcal{M}$ if $M \notin \mathcal{M}$, but all proper minors of M are in $\mathcal{M}$. A minor-closed class of matroids is characterised by its set of excluded minors, but, in general, there may be infinitely many of them. For example, if $\mathbb{F}$ is an infinite field, then the class of $\mathbb{F}$-representable matroids has an infinite number of excluded minors. In striking contrast to this, Tutte proved the following theorem.

Theorem 1. *A matroid M is* GF(2)*-representable if and only if it does not have $U_{2,4}$ as a minor.*

In other words, $U_{2,4}$ is the unique excluded minor for representability over GF(2). Note that

Tutte's theorem resolves Whitney's problem for the class of binary matroids. In 1970 Gian-Carlo Rota conjectured that a similar result held for all finite fields.

Conjecture 2 (Rota's conjecture). *Let $\mathbb{F}$ be a finite field. Then there are only finitely many excluded minors for the class of $\mathbb{F}$-representable matroids.*

In 1979 Bixby and Seymour independently proved Rota's conjecture for GF(3) showing that there were four excluded minors for the class of GF(3)-representable matroids. Some twenty years later, Geelen, Gerards and Kapoor proved that there were seven excluded minors for representability over GF(4). They received the Fulkerson Prize for this work. Jim Geelen reported to me that when he discussed his proof of Rota's conjecture for GF(4) with Bill Tutte, the reply was simply "ah the next case". Deflated though he was at the time, Jim understood that Tutte was, in his inimitable way, pointing out that the constructive techniques that were used in these early cases would not suffice for a resolution of Rota's conjecture in general.

A completely different form of attack was needed. Inspiration came from an Oberwolfach workshop where Paul Seymour described an attempt to generalise some of the results of the Graph Minors Project to binary matroids. Readers familiar with graph minors will already have noticed parallels with Rota's conjecture. Indeed many readers will be familiar with Kuratowski's Theorem which says that the K_5 and $K_{3,3}$ are the only excluded minors for the class of planar graphs. Neil Robertson and Paul Seymour proved that this result is typical. As the culmination of a long series of papers they prove that graphs are well-quasi-ordered under the minor order. Stated in a form suited to this discussion their theorem says the following.

Theorem 3 (Graph WQO Theorem). *Every minor-closed class of graphs has a finite number of excluded minors.*

That sounds a lot like Rota's conjecture. Associated with a graph is a matroid whose elements are the edges of the graph. This matroid is $\mathbb{F}$-representable for any field $\mathbb{F}$. Moreover, minors of graphs and their associated matroids correspond. Robertson and Seymour were aware that the Graph WQO Theorem was likely to be a special case of a more general result for matroids representable over finite fields. After much pain, Jim, Bert and I eventually proved this more general result.

Theorem 4 (Matroid WQO Theorem). *Let $\mathbb{F}$ be a finite field. Then every minor-closed class of $\mathbb{F}$-representable matroids has a finite number of $\mathbb{F}$-representable excluded minors.*

It should be noted that both of the well-quasi-ordering theorems follow — with work — from theorems that give the qualitative structure of members of proper minor-closed classes of graphs or matroids. These theorems are the true workhorses of the projects.

Are we there yet? Definitely not. The Matroid WQO Theorem is about $\mathbb{F}$-representable excluded minors and Rota's conjecture says something quite different. But we have powerful tools and techniques and are well on the way. Space forbids a discussion of the other ingredients necessary to arrive at the final destination of a proof of Rota's conjecture.

Finally I note that it will be some time before a proof of Rota's conjecture appears. The task of writing up our results is lengthy, will require numerous papers, and will take several years.

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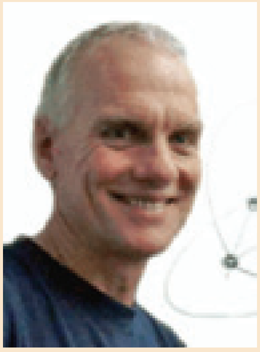

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