On Some Non-Hermitian Ensembles of Random Gaussian Matrices*

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In the broadest sense, a random matrix is merely a matrix with random numbers in it. In practice these matrices often have some symmetry or unitarity conditions, and the entries are frequently drawn from a Gaussian distribution. We call the set of matrices having such-and-such definition an *ensemble*. The main focus of the field of random matrix theory is to analyse the eigenvalue distribution of the ensemble, although the behaviour of the eigenvectors may also be of interest. It may be expected *a priori* that the eigenvalues of a random matrix are scattered uniformly at random over their support (exhibiting the "clumpy" patterns typical of such data), however this is far from true and they instead display strongly correlated behaviour. The most striking instance of this arises from the ubiquitous appearance of the product $\prod_{j<k} |\lambda_k - \lambda_j|^\beta$ in the eigenvalue joint probability density function (jpdf), which is the required Jacobian when changing variables from the matrix entries to the matrix eigenvalues. (We will discuss the β parameter below.) Since this product implies that there is a low probability of finding two eigenvalues close to each other, the product is interpreted as eigenvalue repulsion. This repulsion is found in both Hermitian and non-Hermitian matrix ensembles. Another effect of the Vandermonde product is that, by Vandermonde's well-known identity, the eigenvalue jpdf can be rewritten as a determinant. This latter fact is key to the analysis, allowing the powerful tools of linear algebra to be brought to bear on the problem.

Random Hermitian Gaussian matrices found some early applications in physics, largely due to the work of Eugene Wigner. The problem being faced at the time was the analysis of the highly excited states of heavy nuclei. Modelling the problem as a set of interacting particles quickly leads to a set of unwieldy coupled equations. Rather, Wigner [10] suggested that a statistical approach might be more useful, and he conjectured that the distribution of the spacing between energy levels will be well approximated by the eigenvalue spacing distribution of large, real symmetric, Gaussian matrices (this statement became known as the *Wigner surmise*); this expectation was also proposed by Landau and Smorodinsky [7, Lecture 7]. A few years later, it was comprehensively demonstrated that the repulsive nature of the energy levels could indeed be modelled by eigenvalues of symmetric matrices, and that the results matched Wigner's predictions. Building on this work, Dyson [2] found that these random matrices naturally fell into three classes: real symmetric, complex Hermitian and real quaternion self-dual; a result which is fundamentally due to a theorem of Frobenius that there are exactly three associative division algebras over the reals. It turns out that the parameter β in the Vandermonde product above corresponds to the number of independent real components in these matrix entries: $\beta = 1$ for real ensembles, $\beta = 2$ for complexes and $\beta = 4$ for real quaternions. These values for β have become known as the *Dyson indices*. The ensembles corresponding to these values of β are called the Gaussian orthogonal (GOE), unitary (GUE) and symplectic (GSE) ensembles respectively, due to the invariance properties of their density functions.

Analogous ensembles (which have come to be known as *Ginibre ensembles*) of non-symmetric real, non-Hermitian complex and non-self-dual real quaternion matrices (with Gaussian entries) were introduced in [4], with matching Dyson indices. However, the lack of unitary diagonalisability caused significant complications for the real case, meaning that analytical progress was delayed until the early 1990s. For these non-Hermitian matrices, it is now standard practice to sacrifice orthogonal/unitary diagonalisability for

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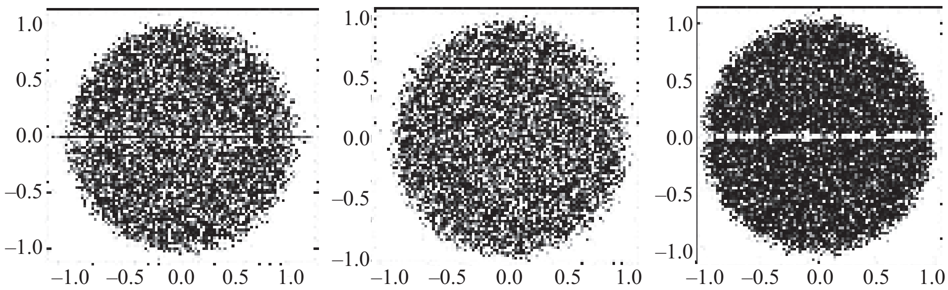

Fig. 1. From left to right we have eigenvalue plots for 120 independent 100×100 Ginibre matrices for $\beta = 1$ (real), 2 (complex), 4 (real quaternion).

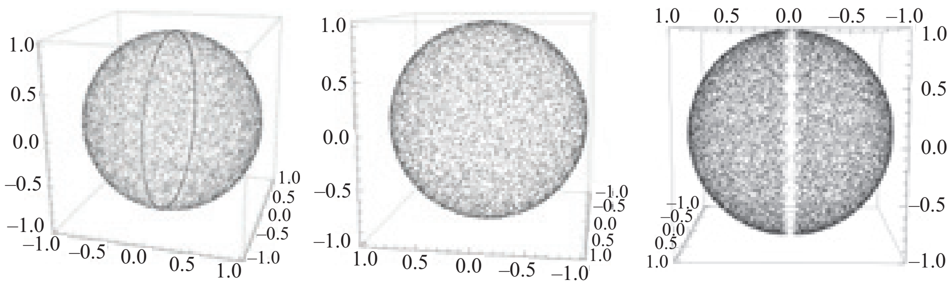

Fig. 2. From left to right we have stereographic eigenvalue plots for 120 independent 100×100 spherical matrices for $\beta = 1$ (real), 2 (complex), 4 (real quaternion).

orthogonal/unitary *triangulisability* by using the Schur decomposition. Whereas the eigenvalues of Hermitian matrices are all real, the eigenvalues of non-Hermitian matrices are (in general) complex: $N \times N$ complex matrices have N independent complex eigenvalues; $2N \times 2N$ real quaternion matrices have N complex-conjugate pairs of eigenvalues; and $N \times N$ real matrices have k real eigenvalues and $(N-k)/2$ complex-conjugate pairs. Further, given the eigenvalue repulsion implied by the Vandermonde product, we note that the complex-conjugate pairs in the real and real quaternion ensembles result in an effective repulsion of eigenvalues from the real axis (except for the strictly real eigenvalues in the real ensembles). This is demonstrated in the simulations of Fig. 1. Note that (except for near the real line) the eigenvalues are roughly uniformly distributed on a disk, which, before scaling is of radius proportional to $\sqrt{N}$. This is a consequence of a universality result — called the *circular law* — which says that in the limit of large matrix dimension the eigenvalue density, suitably scaled, of any matrix with independently and identically distributed (iid) entries (assuming finite mean and unit variance) is uniform on the unit disk [9]. The analogous result for Hermitian ensembles is the *semi-circular law*: that the (scaled) eigenvalue density is a semi-circle of unit radius [1].

If we now take two matrices A, B from the same Ginibre ensemble and form the product $Y = A^{-1}B$, then we have formed a so-called *spherical ensemble*.[a] To see the origin of the name, we stereographically project the eigenvalues of simulations of real, complex and real quaternion spherical ensembles in Fig. 2. The complex ensemble yields eigenvalues that are uniformly distributed over the sphere; the real ensemble has a clear ring of eigenvalues on a great circle corresponding to the real line; and the eigenvalue density for the real quaternion ensemble splits into two disjoint hemispheres along the same great circle corresponding to the real line. We say that these matrices are in the spherical universality class because of the *spherical law*, which is analogous to the circular and semi-circular laws discussed above. It states that for two matrices with iid entries, having mean zero and finite variance (in other words, obeying the circular law), then in the limit of large matrix dimension the eigenvalues of the product $Y = A^{-1}B$ are uniformly distributed on the sphere (after stereographic projection).

The eigenvalue correlation functions for the complex ($\beta = 2$) case were studied in [6] and the real ($\beta = 1$) case in [3]. The work done during

[a]The eigenvalues of $Y = A^{-1}B$ are equivalent to the *generalised eigenvalues* of A and B, which are solutions to $\det(B - \lambda A) = 0$.

this Fellowship, and contained in [8], was on the calculation of the correlation functions for the ensemble corresponding to the last of the Dyson indices, real quaternion ($\beta = 4$). The procedure first involves changing variables from the elements of A and B (each an $N \times N$ matrix with Gaussian real quaternion entries) to the elements of $Y = A^{-1}B$, which entails integrating out $4N^2$ degrees of freedom; then further changing variables to the eigenvalues of Y, integrating out another $4N(N-1)$ variables, leaving just the $2N$ complex eigenvalues. Next, a Pfaffian[b] form of the ensemble average is written down and we find polynomials (called *skew-orthogonal polynomials*) that will skew-diagonalise the matrix in the Pfaffian. Armed with these polynomials we are then able to obtain the eigenvalue correlation functions by functional differentiation of the ensemble average. Lastly, we take various limits to find that the eigenvalues are indeed uniformly distributed over the sphere for large N (a consequence of the spherical law), and further, that by zooming in on the sphere near the great circle corresponding to the real line, we recover the same correlation functions as in the real Ginibre ensemble [5].

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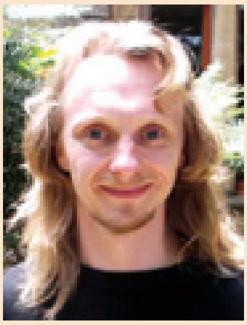

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After growing up in the suburbs of Melbourne, Anthony Mays completed a Bachelor degree in economics and Mandarin Chinese at Monash University. He then undertook a second degree in mathematics and physics before moving to Melbourne University to graduate with honours in mathematics. Continuing on at Melbourne University with a PhD under Peter Forrester, he studied the eigenvalue correlation functions of various random matrix ensembles. The title of his doctoral thesis was A geometrical triumvirate of real random matrices, which he submitted in 2011. Since then, he has been a research assistant at Vanderbilt University, and has been awarded a LiftOff Fellowship by the Australian Mathematical Society. He is currently a postdoctoral researcher at Supélec in France. Anthony is also keen juggler and has a strong interest in the mathematics of juggling — the title of his Honours thesis was Combinatorial aspects of juggling, for which he received the Wyselaskie Scholarship in Mathematics.

[b] For an even dimensional matrix X, which is skew-Hermitian ($X^\dagger = -X$), $\mathrm{Pf}(X) = \sqrt{\det X}$.