Non-commutative L^p-spaces and Analysis on Quantum Spaces

Hun Hee Lee

1. Classical L^p-space and Fourier Analysis

Most of the people with a background of real analysis from undergraduate or graduate courses must be familiar with the concept of L^p-spaces. For a given measure space (X, Σ, μ) we define $L^p(X, \mu)$ as follows.

$$\begin{cases} L^p(X,\mu) = \{f : X \to \mathbb{C} \mid |f|^p \text{ is integrable}\}, 1 \le p < \infty, \\ L^\infty(X,\mu) = \{f : X \to \mathbb{C} \mid f \text{ is essentially bounded.}\} \end{cases}$$

The space $L^p(X, \mu)$ is endowed with the following p-norm.

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}}, \ 1 \le p < \infty, \ \ \|f\|_\infty = \|f\|_{\text{ess-sup}}.$$

Here, $\|\cdot\|_{\text{ess-sup}}$ refers to the essential supremum norm. In particular, for a counting measure μ we denote $L^p(X, \mu)$ by $\ell^p(X)$.

L^p-spaces are Banach spaces with respect to the p-norm and are essential tools for any kind of analysis. Among those L^p-spaces the cases of $p = 1, 2, \infty$ are, of course, in special positions. We understand L^2-spaces as Hilbert spaces with inner product, and L^∞-spaces are commutative von Neumann algebras with respect to pointwise multiplication. Lastly, L^1-space is the predual space of L^∞-space. These L^p-spaces are closely related to various analysis problems, and we can find good examples from Fourier analysis. Now we consider Fourier transform $\mathcal{F}$ defined on the circle group $\mathbb{T}$. For an integrable function $f : \mathbb{T} \to \mathbb{C}$ we define its Fourier transform $\mathcal{F}f$ as a sequence $\mathcal{F}f = (\mathcal{F}f(n))_{n\in\mathbb{Z}}$ indexed by $\mathbb{Z}$, where

$$\mathcal{F}f(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-\text{int}}\,dt.$$

Fourier transform $\mathcal{F}$ behaves according to L^p-space. The Plancherel theorem says

$$\mathcal{F} : L^2\left(\mathbb{T}, \frac{1}{2\pi}dt\right) \to \ell^2(\mathbb{Z})$$

is an onto isometry, and the Hausdorff–Young inequality says for $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{q} = 1$ the map

$$\mathcal{F} : L^p\left(\mathbb{T}, \frac{1}{2\pi}dt\right) \to \ell^q(\mathbb{Z})$$

is a contraction (i.e. operator norm decreasing). The above exploits two important examples of L^p-spaces, namely L^p-spaces coming from probability measures and the ones coming from discrete measures.

Non-commutative L^p-spaces are generalisations of classical L^p-spaces in a non-commutative way. In this article we would like to give a very gentle introduction of non-commutative L^p-spaces focusing on a few simple cases and consider applications to quantum spaces. Here, the term "quantum space" is not defined rigorously, but implies well-known examples such as locally compact quantum groups and non-commutative torus. These quantum spaces are described in operator algebras such as C^*-algebras and von Neumann algebras. In order to consider analysis problems on those spaces we need some kind of function spaces different from classical L^p-spaces. This role can be taken by non-commutative L^p-spaces, which we will see in the sequel.

2. Non-commutative L^p-spaces

Let us take a look back of classical L^p-spaces. When a measure space (X, Σ, μ) is given the algebra $L^\infty(X, \mu)$ of essentially bounded measurable functions can be understood as an algebra of operators via the following isometric embedding.

$$\Phi : L^\infty(X,\mu) \to B(L^2(X,\mu)), \ \ f \mapsto M_f.$$

Here, $B(H)$ means the algebra of bounded linear map acting on a Hilbert space H and M_f is the multiplication operator with respect to f given by $M_f(g) = fg$, $g \in L^2(X, \mu)$. Thus, $L^\infty(X, \mu)$ is a commutative subalgebra of $B(L^2(X, \mu))$ closed under taking adjoint. Moreover, it contains the identity operator and closed under a certain topology called the weak operator topology. We call a subalgebra of $B(H)$ satisfying all the properties we mentioned about $L^\infty(X, \mu)$ except commutativity, a von Neumann algebra. Now we need a special functional $\varphi : f \mapsto \int_X f\,d\mu$ on $L^\infty(X, \mu)$ in order to construct $L^p(X,\mu)$. Note that this functional is, in general, not defined on the whole algebra. Thus, it is natural to expect a functional ϕ on a von Neumann algebra $\mathcal{M}$ which will play the same role of φ. For simplicity we will consider the following two cases.

2.1. Schatten class

The first example is the case $\mathcal{M} = B(H)$. In this case we may take ϕ to be the trace Tr. Here, Tr is the functional defined by $\mathrm{Tr}X = \sum_i \langle Xh_i, h_i\rangle$, $X \in B(H)$, for an orthonormal basis $\{h_i\}_{i\in I}$ of H. Note that when H is infinite dimensional it is not defined on the whole $B(H)$. Now, the corresponding non-commutative L^p-space $L^p(\mathcal{M})$, $1 \leq p < \infty$ is defined by

$$L^p(B(H)) = \{X \in B(H) : \mathrm{Tr}(|X|^p)^{\frac{1}{p}} < \infty\}.$$

The space $L^p(B(H))$ is called the Schatten class on H and we usually denote it by $S^p(H)$. This case is a non-commutative counterpart of a sequential $\ell^p(X)$-space coming from discrete measures. In particular, when $\dim H = n < \infty$, the spaces $B(H)$ and $S^p(H)$ are usually denoted by M_n and S^p_n, which are nothing but the space of all $n \times n$ complex matrices as sets.

2.2. Non-commutative (W^*-) probability space

When there is a special functional (i.e. normal, faithful and tracial state) ϕ on a von Neumann algebra $\mathcal{M}$ we call $(\mathcal{M}, \phi)$ a non-commutative (W^*-) probability space. Here, ϕ being state means that ϕ is linear, $\phi(X^*X) \geq 0$, $X \in \mathcal{M}$ and $\phi(I) = 1$. Moreover, ϕ being tracial means $\phi(XY) = \phi(YX)$, $X, Y \in \mathcal{M}$. In this case ϕ is defined on the whole algebra $\mathcal{M}$ and the p-norm can be defined on $\mathcal{M}$ by $\|X\|_p := \phi(|X|^p)^{\frac{1}{p}}$, $X \in \mathcal{M}$. Now the corresponding non-commutative L^p-space $L^p(\mathcal{M}, \phi)$ is defined as the completion of $(\mathcal{M}, \|\cdot\|_p)$. This case a non-commutative counterpart of the classical L^p-spaces coming from probability measures.

The above kind of von Neumann algebra can be produced out of any discrete group G. Let $\lambda : G \to B(\ell^2(G))$, $\lambda(x)f(y) = f(x^{-1}y)$, $f \in \ell^2(G)$, $x, y \in G$ be the left regular representation of G. Then, the group von Neumann algebra $VN(G)$ generated by the set of operators $\{\lambda(x) : x \in G\}$ is equipped with a natural normal, faithful, and tracial state ϕ given by $\phi(\sum_{x\in G} a_x\lambda(x)) = a_e$, i.e. the functional of evaluating the coefficient of the identity if we regard $(a_x)_{x\in G}$ as the "Fourier coefficient" of the operator $\sum_{x\in G} a_x\lambda(x)$.

2.3. General case

There are many von Neumann algebras without such a good tracial functional, which we call type III von Neumann algebras. A definition of non-commutative L^p-spaces that also can be applied to type III cases by U Haagerup in 80's ([4]).

The general non-commutative L^p-spaces share many properties of classical L^p-spaces. For example, Hölder inequality, the duality between L^p and L^q for $1/p + 1/q = 1$, and complex interpolation still hold in this non-commutative context ([7]). Of course, not everything is good as before, for example, non-commutative L^p-spaces are no more lattices with respect to the natural order structure.

3. Analysis on Quantum Spaces and Non-commutative L^p-spaces

In the following we will take a look at a few cases of L^p-analysis problems on quantum space using non-commutative L^p-spaces.

3.1. Fourier analysis on the dual of locally compact groups

For a locally compact group G we can similarly get the group von Neumann algebra $VN(G)$ as in 2.2. The non-commutative L^1-space $L^1(VN(G))$ associated to $VN(G)$ can be understood as a subspace of $C_0(G)$ (the space of continuous functions on G vanishing at infinity) and is known to be a commutative Banach algebra with respect to pointwise multiplication. The space $L^1(VN(G))$ is called the *Fourier algebra* on G and usually denoted by $A(G)$. The Fourier algebra is an important object in abstract harmonic analysis reflecting many properties of the underlying group. For example, the spectrum, $\mathrm{spec}A(G)$, of G is homeomorphic to G itself, and it is known that $A(G)$ is amenable as a Banach algebra together with an operator space structure if and only if G is amenable as a locally compact group ([8]). Recently, there has been an investigation on the relationship between the homological properties of $L^p(VN(G))$ as a left module of $A(G)$ and the properties of G as a locally compact group ([3]).

In general, the Banach space structure of $L^p(VN(G))$ is complicated, but the case of compact G is relatively simple. From the representation theory of compact groups it is well-known that $VN(G)$ is isomorphic to a direct sum of matrix algebras

$$VN(G) \cong \bigoplus_{\pi \in \widehat{G}} M_{d_\pi}.$$

The associated non-commutative L^p-space is the ℓ^p-direct sum of the corresponding Schatten classes

$$\ell^p\text{-}\bigoplus_{\pi \in \widehat{G}} S^p_{d_\pi}.$$

Here, $\widehat{G}$ is the collection of irreducible unitary representations of G (actually, their equivalence classes) and d_π is the dimension of $\pi \in \widehat{G}$.

On the other hand, the case of discrete groups gets a lot of attention of operator algebraists, especially the case of free group $\mathbb{F}_n$ with $n \geq 2$ generators. A recent research about the Laplacian coming from the natural length function on $\mathbb{F}_n$ and their Riesz transform is notable ([5]).

3.2. Fourier analysis on non-commutative torus

Very recently, Chen/Xu/Yin ([2]) successfully transferred many Fourier analysis problems holding on d-dimensional torus $\mathbb{T}^d$, $d \geq 2$ to the setting of non-commutative d-torus $\mathbb{T}^d_\theta$. When $d = 2$ for an irrational number θ the non-commutative 2-torus $\mathbb{T}^2_\theta$ is the universal C^*-algebra generated by two unitaries U, V satisfying the commutation relation $UV = e^{2\pi i\theta}VU$. On $\mathbb{T}^2_\theta$ we have a natural (faithful) tracial state ϕ given by

$$\phi\left(\sum_{n,m\in\mathbb{Z}} a_{n,m} U^n V^m\right) = a_{0,0}.$$

We denote the von Neumann algebra generated by $\mathbb{T}^2_\theta$ in the GNS-representation of ϕ by $L^\infty(\mathbb{T}^2_\theta)$. Chen/Xu/Yin considered several topics like mean convergence of Fourier series and maximal inequality on the corresponding non-commutative L^p-spaces $L^p(\mathbb{T}^2_\theta)$. Moreover, they showed that Fourier multipliers on $L^p(\mathbb{T}^2_\theta)$ are essentially the same as the ones on $L^p(\mathbb{T}^2)$ when we restrict our attention to completely bounded multipliers.

3.3. Hypercontractivity on non-commutative probability spaces

The free von Neumann algebra is another example of non-commutative (W^*-) probability space. The sum of the left creation operator and the left annihilation operator on the free Fock space is called the free gaussian. The free gaussians generate the free von Neumann algebra, which is known to be isomorphic to the group von Neumann algebra with respect to free groups. On the free von Neumann algebras there a tracial state coming from the vacuum vector, so that we get a non-commutative (W^*-) probability space. As in the classical case we can define free Ornstein–Uhlenbeck semigroup P_t, so that we are naturally interested in its hypercontractivity problem. P_t is called hypercontractive if there is $t_{p,q} > 0$ such that $P_t : L^p \to L^q$, $1 < p \leq q < \infty$ is a contraction for all $t \geq t_{p,q}$. The solution of this problem has been given by P Biane ([1]), and the problem has been extended to type III case in [6].

4. Epilogue

Quantum spaces are getting more and more attentions from various fields of mathematics and physics. So far, researches on quantum spaces have been focusing mainly on algebraic aspects and less on analytic aspects. This is partly because the objects themselves are described in (operator) algebraic language and the tools for non-commutative analysis was not available in the beginning. However, in the past 15 years the understanding of non-commutative L^p-spaces have been extensively increased by the researchers including G Pisier, M Junge and Q Xu, which allowed us to answer some of the analysis problems on quantum spaces. The list of unsolved problems in the quantum setting is a long one compared to the classical situation. We finish this short article with a recommendation of the survey paper [7] Pisier/Xu for the readers interested in more details of non-commutative L^p-spaces.

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Hun Hee Lee

Seoul National University, Korea
hunheelee@snu.ac.kr

Hun Hee Lee was born in Daejeon, Korea on Feb 17, 1976. He obtained his bachelor (1997), master (1999) and PhD (2004) degree from KAIST (Korea Advanced Institute of Science and Technology) located in Daejeon, Korea. He spent 4 years at the department of Mathematics, Chungbuk National University, Korea as an assistant professor and moved to Seoul National University in March 2013.

Lee's interests on the research are mainly divided into two parts. The first one concerns about abstract harmonic analysis focusing on the Fourier algebra of a locally compact group, and the second one concerns about non-commutative probability using operator spaces and non-commutative Lp-spaces. In a near future he hopes to find a way to make a bridge between two of his main interests using the language of locally compact quantum groups.

In 2011 he has been chosen as a TJ park Junior Faculty Fellowship based on his recent achievements and in 2012 his research has been chosen to be one of the 50 outstanding results in science of the year.