What is Hyperbolic Geometry?*

Mahan Mj

Abstract. We give a brief introduction to hyperbolic geometry, including its genesis. Some familiarity with high school calculus and co-ordinate geometry is all that is assumed and most of the article should, in principle, be accessible to beginning undergraduates.

1. Introduction

1.1. History

Euclid was the first to formalise geometry into an axiomatic system. One of his axioms called the parallel postulate has been the focus of a lot of mathematical attention and work for almost two millennia. It states:
Given a straight line L and a point x outside it, there exists a unique straight line L' passing through x and parallel to L.

This postulate has certainly existed from at least as far back as 200 BC and much effort went in to try to prove it from the other axioms of Euclid. In ancient times, Proclus, Omar Khayyam, Witelo, Gersonides, Alfonso, amongst others made the attempt. In more recent times, Saccheri, Wallis, Lambert, and even Legendre failed in this attempt, with good reason as we hope to show in this article.

To appreciate what this problem means, we first state Euclid's axioms (appearing in the first book of *Elements*).

(1) A straight line may be drawn from any point to any other point.
(2) A finite straight line may be extended continuously in a straight line.
(3) A circle may be drawn with any centre and any radius.
(4) All right angles are equal.
(5) If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if extended indefinitely, meet on the side on which the angles are less than two right angles.

It is the fifth postulate that is equivalent to the parallel postulate stated above. The reason why people tried to prove it from the rest of the axioms is that they thought it was not "sufficiently self-evident" to be given the status of an axiom, and an "axiom" in Euclid's times was a "self-evident truth". Almost two millennia passed with several people trying to prove the fifth postulate and failing.

Gauss started thinking of parallels about 1792. In an 1824 letter to F A Taurinus, he wrote: "The assumption that the sum of the three angles (of a triangle) is smaller than two right angles leads to a geometry which is quite different from our (Euclidean) geometry, but which is in itself completely consistent." But Gauss did not publish his work. Already in the 18th century, Johann Heinrich Lambert introduced what are today called hyperbolic functions and computed the area of a hyperbolic triangle. In the 19th century, hyperbolic geometry was extensively explored by the Hungarian mathematician Janos Bolyai and the Russian mathematician Nikolai Ivanovich Lobachevsky, after whom it is sometimes named. Lobachevsky published a paper entitled *On the principles of geometry* in 1829–1830, while Bolyai discovered hyperbolic geometry and published his independent account of non-Euclidean geometry in the paper *The absolute science of space* in 1832. The term "hyperbolic geometry" was introduced by Felix Klein in 1871. See [2] and [4] for further details on history.

To get back to our article, we restate the parallel postulate, expanding it somewhat and underscore the terms we will investigate.

Postulate 1.1. (Parallel Postulate) *Given a* **straight line** *L in a* **plane** *P and a point x on the plane P lying outside the line L, there exists a unique straight line L' lying on P passing through x and* **parallel** *to L.*

Then the problem we address is:

Problem 1.2. *Prove the Parallel Postulate 1.1 from the other axioms of Euclidean geometry.*

To properly appreciate the (rather unexpected) solution to Problem 1.2 we need to investigate the following keywords more thoroughly:

(1) straight line
(2) plane
(3) parallel.

1.2. Terms and definitions

Definition 1.3. The Euclidean plane is $\mathbb{R}^2$

*This is an expanded and revised version of an article with the same title that appeared in the "Ramanujan Mathematical Society Newsletter" Vol. 19 No. 2, September 2009. We are grateful to the Ramanujan Mathematical Society for allowing us to submit the revised version to APMN.

equipped with the metric

$$ds^2 = dx^2 + dy^2.$$

We have suddenly sprung the new concept of a metric on the unsuspecting reader. A word of clarification is necessary to justify our assurance in the Abstract that some familiarity with high school calculus and co-ordinate geometry is all that is assumed. We all know that distances in the Cartesian plane are measured by the formula

$$d((x_1, y_1), (x_2, y_2)) = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{\frac{1}{2}}$$

for points $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$.

Definition 1.3 is just the infinitesimal form of this formula. The reason why it is called a metric is that it provides us a means of measuring lengths of curves σ (thought of as *smooth* maps of $[0, 1]$ into $\mathbb{R}^2$) according to the formula

$$l(\sigma) = \int_0^1 ds = \int_0^1 \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{\frac{1}{2}} dt \qquad \text{(A)}$$

for some parametrisation $x = x(t), y = y(t)$ of the curve σ.

We now turn to the notion of a straight line in this context. A fact we are quite familiar with, but whose proof is not quite trivial, is the following.

Proposition 1.4. *Given two points $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, the straight line segment between (x_1, y_1) and (x_2, y_2) is the unique path that realises the shortest distance (as per formula A) between them.*

We shall not prove this here, but when we turn to hyperbolic geometry we shall give the proof of a corresponding statement in hyperbolic geometry and leave the reader to modify it appropriately to prove Proposition 1.4. Note that the term **straight line segment** in Proposition 1.4 means **straight line** in the sense of Euclid.

Finally we give the precise meaning of the word *parallel*.

Definition 1.5. Two bi-infinite straight lines are said to be parallel if they do not intersect.

Here, by a bi-infinite straight line we mean the result of extending a straight line segment infinitely in either direction, so that any subsegment of it is a straight line.

Note: All the notions above boil down to the single notion of a metric in Definition 1.3. It is striking that all the features of Euclidean geometry have this single structural definition at their foundation. In the language of logic, $\mathbb{R}^2$ equipped with the metric

$$ds^2 = dx^2 + dy^2$$

is called a *model* for Euclidean geometry. We shall not go into this, though the real solution to Problem 1.2 involves at least an implicit investigation of these foundational issues.

2. Metric Geometry

In this article, we shall be interested in a somewhat more general form of a metric than Definition 1.3. This will give us quite a general class of "models". We state this somewhat informally below.

Definition 2.1. A metric on an open subset U of the Euclidean plane $\mathbb{R}^2$ is a method of computing arc lengths $l(\sigma)$ as per the formula

$$ds^2 = f(x, y)dx^2 + g(x, y)dy^2.$$

Thus the length of σ will be given by the formula

$$l(\sigma) = \int_0^1 ds = \int_0^1 \left[f(x(t), y(t))\left(\frac{dx}{dt}\right)^2 + g(x(t), y(t))\left(\frac{dy}{dt}\right)^2\right]^{\frac{1}{2}} dt \qquad \text{(B)}$$

for some parametrisation $x = x(t), y = y(t)$ of the curve σ.

An **isometry** I is a map that preserves the metric, i.e. if $I((x, y)) = (x_1, y_1)$ then

$$f(x, y)dx^2 + g(x, y)dy^2 = f(x_1, y_1)dx_1^2 + g(x_1, y_1)dy_1^2.$$

The property of straight lines given by Proposition 1.4 is then abstracted out to give the next definition.

Definition 2.2. An arc σ in U equipped with the metric above is said to be a geodesic if it locally minimises distances, i.e. there exists $\epsilon > 0$ such that for $a, b \in \sigma([0, 1])$, if there is some arc of length at most ϵ connecting a, b, then the subarc of σ joining a, b is the unique path that realises the shortest distance (as per formula (B)) between them.

The definition of parallel lines 2.3 goes through in this more general context by just replacing the term *straight line* by the term *geodesic*.

Definition 2.3. Two bi-infinite geodesics are said to be parallel if they do not intersect.

3. Hyperbolic Geometry

3.1. The model

A Model for hyperbolic geometry is the upper half plane $\mathbf{H} = (x, y) \in \mathbb{R}^2, y > 0$ equipped with the metric

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2). \qquad \text{(C)}$$

H is called the Poincare upper half plane in honour of the great French mathematician who discovered it.

Motivation, an aside: Without any motivation, the model described above seems to have come out of thin air. This is the only place where, during the course of this article, some more mathematical background is necessary to fully appreciate what is going on. The young reader is therefore encouraged to take the above model at face value and skip this motivation.

Poincare who proposed the above model, came to it from complex analysis. It is a fact that any simply connected complex 1-manifold (whatever that means) is essentially one of the following three:

(1) The Riemann sphere $\hat{\mathbb{C}}$
(2) The complex plane $\mathbb{C}$
(3) The upper half plane $\mathbf{H} = \{z \in \mathbb{C} : Im(z) > 0\}$.

Poincare came to hyperbolic geometry from complex analysis and, suffice to say, in this context the model for $\mathbf{H}$ is quite natural as one of three possible models for a simply connected complex 1-manifold.

3.2. Geodesics and isometries

It is not hard to see that **vertical straight lines** (given by $x = k$, a constant) are geodesics (see Definition 2.2). Consider two points $a = (k, u)$ and $b = (k, v)$ in $\mathbf{H}$. Then any path σ from a to b that is not vertical will have nonzero $\frac{dx}{dt}$ at some point. Then its length computed as per formula (C) will be strictly greater than that of the path $P(\sigma)$, where P is the projection onto the vertical line $x = k$, given by $P(x, y) = (k, y)$. Hence as per Definition 2.2, we have

Lemma 3.1. *Vertical straight lines in* $\mathbf{H}$ *are geodesics. In fact, the vertical segment between* a, b *is the* **unique** *geodesic between* a, b.

Now, we compute some explicit isometries (see Definition 2.2).

Lemma 3.2. *Define* $f : \mathbf{H} \to \mathbf{H}$ *by* $f(x, y) = (x + a, y)$ *for some fixed* $a \in \mathbb{R}$.
Define $g : \mathbf{H} \to \mathbf{H}$ *by* $g(z) = \frac{R^2}{\bar{z}}$ *for some fixed* $R > 0$, *where* $\bar{z}$ *denotes the complex conjugate of* z.
Then f, g *are isometries of* $\mathbf{H}$.

Proof. First note that f is given in real co-ordinates and g is given in complex co-ordinates. To check that f is an isometry, put $x_1 = x + a$, $y_1 = y$. Then (since a is a constant) $\frac{1}{y^2}(dx^2 + dy^2) = \frac{1}{y_1^2}(dx_1^2 + dy_1^2)$ and hence f preserves the metric given by formula (C), i.e. f is an isometry. □

Next, to show that g is an isometry, we express formula (C) in complex co-ordinates as

$$ds^2 = \frac{dz d\bar{z}}{Im(z)^2}.$$

It is a simple calculation to show that if we put $z_1 = g(z) = \frac{R^2}{\bar{z}}$, then $\frac{dz d\bar{z}}{Im(z)^2} = \frac{dz_1 d\bar{z}_1}{Im(z_1)^2}$. This shows that g is an isometry. □

Maps such as f are called *parabolic translations* and those like g are called *inversions*. The map g is an inversion about a semi-circle of radius R centred at 0. A more geometric description of g is as follows. Take a circle of radius R centred at 0. Then every point in $\mathbf{H}$ lies on a unique ray through 0. Let $p \in \mathbf{H}$ be on some such ray r. Then $g(p)$ is the unique point on r for which the product of the radial co-ordinates of p and $g(p)$ is R^2.

We shall have need only for inversions in what follows. These maps are the hyperbolic geometry analogues of **reflections** in plane Euclidean geometry.

There is nothing special about the point 0 in Lemma 3.2 for the map g. We might as well shift the origin 0 to a point p on the real axis and consider inversions about semi-circles centred at p. Then essentially the same computation shows that inversions about semi-circles of arbitrary radius R centred on the real line are isometries. The reader may refer to [1] and [3] for similar computations.

Next, from Definition 2.2, it follows that the image of a geodesic under an isometry is another geodesic. Hence images of vertical geodesics under inversions are geodesics.

Now consider a semi-circle C_1 with centre at $(a, 0)$ with radius R. Consider the semi-circle C_2 with centre at $(a + R, 0)$ with radius $2R$. Let inversion in C_2 be denoted by I_2. Let l denote the vertical geodesic $x = (a - R)$. Then it is a simple exercise in Euclidean geometry to show that $I_2(l) = C_1$. We have thus finally established the following.

Theorem 3.3. *In the hyperbolic plane* $\mathbf{H}$, *vertical straight lines and semi-circular arcs with centre on the real axis are geodesics.*

Let $\mathcal{L}$ denote the collection of vertical straight lines and semi-circular arcs with centre on the real axis. Given any $l \in \mathcal{L}$ and any point p outside l on $\mathbf{H}$ there are infinitely many $l' \in \mathcal{L}$ through p such that $l \cap l' = \emptyset$.

We are finally in a position to answer Question 1.2 adequately. There are models ($\mathbf{H}$ in the above description) which satisfy all axioms of Euclidean geometry except the parallel postulate.

Theorem 3.4. *We have the following* **New Parallel Postulate**:
Given a geodesic L *in* $\mathbf{H}$ *and a point* x *on* $\mathbf{H}$ *lying outside* L, *there exist infinitely many bi-infinite geodesics* L' *lying on* $\mathbf{H}$ *passing through* x *and parallel to* L.

The careful reader will notice that we have only violated the **uniqueness** part of the parallel postulate in constructing $\mathbf{H}$. Existence continues to hold.

To end this section we mention the fact that the collection $\mathcal{L}$ described above is the collection of **all** geodesics in $\mathbf{H}$. The interested reader can prove this using the uniqueness part of Lemma 3.1.

4. Fuchsian Groups and Closed Hyperbolic Surfaces

We now come to the study of discrete subgroups of the isometry group of the hyperbolic plane. Poincaré initiated the study of these groups. An octagon with suitable side identifications (see Diagram 1 below) gives a genus two surface topologically.

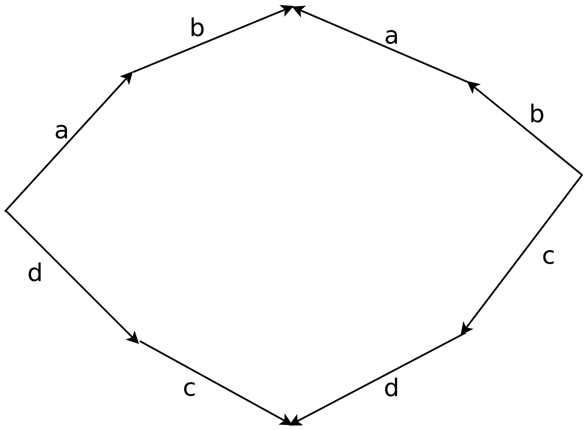

Diagram 1

This is just a topological picture. We want to convert the flabby topological information into a more precise geometric picture. We first suppose that the above octagon is hyperbolic, i.e. it is isometric to a piece of the hyperbolic plane. To glue it as per the recipe given in the Diagram 1 above, we need to ensure two things:

(1) The sides labelled by the same letters are geodesics of equal length.
(2) The angle sum obtained at the vertex to which all the eight vertices of the octagon are identified is equal to 2π.

Towards this, we first consider a more symmetric model of the hyperbolic plane, the Poincaré disk. This is the unit disk in the complex plane with the metric $\frac{4|dz|^2}{(1-|z|^2)^2}$. It turns out that the Poincaré disk is isometric to the upper half plane with the hyperbolic metric.

Now consider an *ideal hyperbolic octagon*, i.e. an octagon, all whose vertices are on the circle at infinity. See Diagram 2.

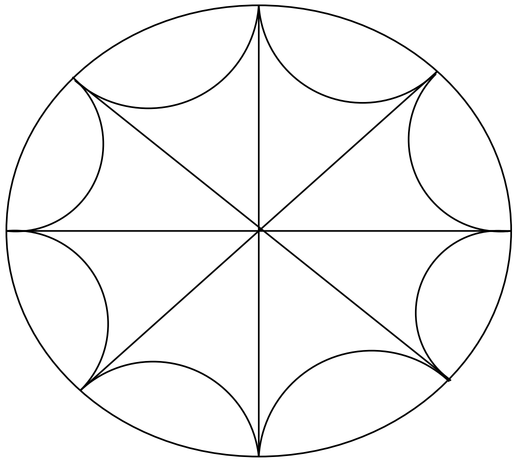

Diagram 2

The geodesics in this model are semi-circles meeting the boundary circle at right angles. Since adjacent sides (geodesics) both meet the boundary circle at right angles, it follows that the internal angles of an ideal octagon are all **zero**. Now start shrinking the octagon slowly towards a very small regular octagon in the Poincaré disk, ensuring that the octagon is symmetric about the origin. Then a very small regular hyperbolic octagon has internal angles very close to the internal angles of a regular Euclidean octagon, which is $\frac{3\pi}{4}$. It follows from the intermediate value theorem that at some intermediate stage, all internal angles equal $\frac{\pi}{4}$. Take the regular hyperbolic octagon with all internal angles equal to $\frac{\pi}{4}$ and glue alternate sides as per the recipe in Diagram 1.

Since all sides are equal, it follows that sides labelled by the same letters are geodesics of equal length. The angle sum obtained at the vertex to which all the eight vertices of the octagon are identified is equal to $8 \times \frac{\pi}{4} = 2\pi$. Thus the identification space is a hyperbolic genus two surface. The isometries of the hyperbolic plane that identify sides labelled by the same letter (preserving orientation) generate a discrete subgroup of the isometry group of the hyperbolic plane, such that the quotient of the hyperbolic plane by this group is the hyperbolic genus two surface.

The above construction is immensely flexible. We can vary lengths of sides and angles ensuring only the conditions that sides labelled by the same letters are geodesics of equal length and that the angle sum obtained at the vertex to which all the eight vertices of the octagon are identified is equal to 2π. Thus we get continuously many hyperbolic genus two surfaces by varying the above construction.

5. Kleinian Groups and Thurston's Work

In one dimension higher, we have hyperbolic 3-space $\mathbb{H}^3 = \{(x, y, z) : z > 0\}$ equipped with the metric $ds^2 = \frac{dx^2+dy^2+dz^2}{z^2}$. Discrete subgroups of the isometry group of hyperbolic 3-space are termed Kleinian Groups. As in the two dimensional case, one can construct an all-right hyperbolic dodecahedron, i.e. a dodecahedron in hyperbolic space such that

(1) All faces are totally geodesic regular hyperbolic pentagons.

(2) All dihedral angles are right angles.

Gluing opposite faces of the dodecahedron by a three-fifths twist we obtain a closed hyperbolic 3-manifold called the Seifert–Weber dodecahedral space. The group of face-pairing transformations generate a Kleinian group such that the quotient of hyperbolic 3-space by this group is the Seifert–Weber dodecahedral space.

Thurston's Geometrisation Conjecture (proved by Perelman) states roughly that a generic topological 3-manifold is hyperbolic. A slightly more precise formulation is that *any* topological 3-manifold can be canonically decomposed along spheres and tori such that each resulting piece carries a geometric structure. Also amongst geometric structures, a hyperbolic structure is generic. The celebrated Poincaré Conjecture is a small component of this conjecture. The first impetus towards this conjecture was given by Thurston himself, who, in his Fields' Medal winning work showed that a vast collection of topological 3-manifolds (the so-called Haken atoroidal ones) carry a hyperbolic structure. We refer the reader to [5] for this, amongst several other conjectures of Thurston. Almost all these conjectures have now been resolved affirmatively after three decades of hard work by a number of mathematicians — a tribute to the great insight that Thurston had.

References

[1] R. Benedetti and C. Petronio, *Lectures on Hyperbolic Geometry*, Springer Universitext (1992).

[2] J. G. Ratcliffe, *Foundations of Hyperbolic Manifolds*, 2nd Edition (GTM Springer, 2005).

[3] G. P. Scott, The geometries of 3-manifolds, *Bull. London Math. Soc.* **15** (1983) 401–487.

[4] J. Stillwell, *Sources in Hyperbolic Geometry*, Volume 10 in AMS/LMS series History of Mathematics (1996).

[5] W. P. Thurston, Three dimensional manifolds, Kleinian groups and hyperbolic geometry, *Bull. Amer. Math. Soc.* (1982) 357–382.

Mahan Mj

RKM Vivekananda University, India

Mahan Mj is a Professor of Mathematics at School of Mathematical Sciences, RKM Vivekananda University.