C S Seshadri — A Glimpse of His Mathematical Personality

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C S Seshadri

Abstract. The aim of the article is to give a quick insight into the mathematical personality of C S Seshadri who turned eighty this year. We take a small journey through the areas of research where he has made outstanding contributions.

1. Introduction

Conjeevaram Srirangachari Seshadri was born on February 29, 1932, in Kanchipuram. He was the eldest among eleven children of his parents, Sri C Srirangachari (a well-known advocate in Chengleput, a town 60 km South of Chennai) and Srimati Chudamani. Seshadri's entire schooling was in Chengleput. He joined the Loyola College, Chennai in 1948, and he graduated from there in 1953 with a BA (Hons) degree in mathematics.

During his years at college, Professor S Narayanan and Fr C Racine played a decisive role in Seshadri's taking up mathematics as a profession.

Seshadri joined the Tata Institute of Fundamental Research, Mumbai in 1953 as a student. He received his PhD degree in 1958 from the Bombay University for his thesis entitled "Generalised multiplicative meromorphic functions on a complex manifold". His thesis adviser was Professor K Chandrasekaran who shaped the mathematical career of Seshadri as he did for many others.

Seshadri spent the years 1957–1960 in Paris, where he came under the influence of many great mathematicians of the French school, like Chevalley, Cartan, Schwartz, Grothendieck and Serre.

He returned to the TIFR in 1960 and was a member of the faculty of the School of Mathematics until 1984, where he was responsible for establishing an active school of algebraic geometry. He moved to the Institute of Mathematical Sciences, Chennai in 1984.

In 1989, Seshadri became the director of the Chennai Mathematical Institute, then called the SPIC Mathematical Institute, founded by A C Muthiah.

Seshadri is a recipient of numerous distinctions. He received the Bhatnagar Prize in 1972 and was elected a fellow of the Royal Society, London in 1988. He has held distinguished positions in various centres of mathematics, all over the world. In 2006, Seshadri was awarded the TWAS Science Prize along with Jacob Palis for his distinguished contributions to science.

In the past five years since he received the National Professorship, Seshadri has been awarded the H K Firodia Award for Excellence in Science and Technology, Pune, 2008, the Rathindra Puraskar from Shantiniketan's Visva-Bharati University, Kolkata, 2008, the Padma Bhushan by the President of India, 2009. More recently, he was elected a Foreign Associate of the US National Academy of Sciences, 2010.

On February 29, 2012, Seshadri turned eighty and the Chennai Mathematical Institute and the Institute of Mathematical Sciences together held their first joint International Mathematics Conference in his honour.

Seshadri is also an accomplished exponent of the Carnatic Music and even to this day he continues to religiously do his musical *sadhana*.

Seshadri married Sundari in 1962, and they have two sons, Narasimhan and Giridhar.

2. Seshadri's Mathematical Work

Over the past fifty years, C S Seshadri has been a towering figure in the mathematical horizon, and his contributions have been central to the development of moduli problems and geometric invariant theory as well as representation theory of algebraic groups. In 2012, his Collected Papers was published in two volumes and runs to nearly 1700 pages. The subject matter in these volumes is a true reflection of the diversity of his mathematical contributions.

J P Serre in his famous paper "Faisceaux algébriques cohérents", posed the following question: "Is a finitely generated projective module over the polynomial ring in several variables free, or equivalently, is an algebraic vector bundle over the n-dimensional affine space trivial?" Seshadri's ingenious solution of Serre's problem on projective modules in two variables was a catalyst for much of the later developments in this area. This work attracted much attention, culminating in the famous Quillen–Suslin theorem. This paper ([1]) was written during Seshadri's stay in Paris in the late 1950's where he came under the influence of Chevalley and Serre. His early work ([3]) on the Picard varieties and related problems has its roots in the Chevalley Seminar where he contributed several important exposés ([2]). Besides the ideas of Chevalley, the construction of the Picard variety of a complete variety uses the descent theory of Cartier for purely inseparable coverings and those related to the existence of a *moduli* for a rational map of a smooth curve into a commutative group variety (in the sense of Rosenlicht). This work was influential in the later work of J P Murre on representability of the Picard functors (*Publications of IHES*, 1964) (see also A Grothendieck's "Fondements de la géometrie algébrique" for these developments).

Subsequently, he took up the problem of constructing "orbit spaces" when one is given a *good* action of a group variety on an algebraic variety. The orbit space X/G in general need not exist as an algebraic variety even when G is a finite group, unlike the complex analytic or differentiable cases. Seshadri showed in [4] that if X is a *normal variety* the obstruction to the existence of an algebro-geometric orbit space comes from a finite group action. This result (see also [11]) is a sort of precursor to the existence of X/G as an *algebraic space* in the sense of M Artin (see also the work of János Kollár, *Annals of Mathematics*, 1997). If moreover G happens to be an abelian variety, Seshadri showed ([5]) that the orbit space always existed as an algebraic variety. An interesting point of this work is that it has a criterion for a Weil divisor on the product of a normal variety and a smooth variety to be locally principal. This led to a stronger version known as the Ramanujam–Samuel theorem which figures in the Appendix to this paper by C P Ramanujam ([5]).

It was around this time in the early 1960s that D Mumford had come up with his deep work on geometric invariant theory and at much about the same time Seshadri and Narasimhan began their work on vector bundles which had its origins in the paper of Weil written in 1938 entitled "Généralisation des fonctions abéliennes".

One of the important developments in algebraic geometry in the last few decades is that of the deep study of moduli problems, starting initially with that of curves, abelian varieties and vector bundles on curves. Initial results on vector bundles on curves were those of Grothendieck on $\mathbf{P}^1$ and Atiyah on elliptic curves. The work of Weil mentioned above contained many ideas on the characterisation and classification of vector bundles on compact Riemann surfaces and their relationship with representations of the fundamental group of the Riemann surface. In 1962, in his talk at the International Congress of Mathematicians, David Mumford gave a sketch of his "Geometric invariant theory", or GIT as it is called now. In this talk, Mumford outlined how GIT could be used to solve moduli problems of curves, abelian varieties and vector bundles. The notions of stability and semistability were introduced in this work of Mumford. He also sketched a proof of the quasi-projectivity of the moduli space of stable bundles of fixed rank and degree.

The theme in the work of Narasimhan and Seshadri ([6, 7]) is the study of the space which parametrises conjugacy classes of representations of the fundamental groups of Riemann surfaces. This can be seen as a non-abelian generalisation of the classical Jacobian of a Riemann surface. In the classical abelian case, the Abel–Jacobi map identifies the space of representations $H^1(X, U(1))$ with the Jacobian of the curve X. The Jacobian can be seen geometrically as the moduli space of holomorphic line bundles on X of degree 0. The corresponding non-abelian analogy is the theme in the papers of Narasimhan–Seshadri.

The basic object in the non-abelian theory turned out to be that of a *stable bundle* obtained in an altogether different context by D Mumford. The paper of Narasimhan–Seshadri studies the space $M(n, d)^s$ of isomorphism classes of *stable holomorphic vector bundles of rank n and degree d*. They prove that $M(n, d)^s$ can be identified as a topological space with the space of *irreducible unitary representations* of $\pi_1(X)$. The main theme of this work can be described as establishing a functorial correspondence between the categories of *irreducible unitary representations of certain Fuchsian groups* and *stable holomorphic bundles of degree d*.

The Narasimhan–Seshadri theorem has had a profound impact on the subject. It has been developed and generalised on many fronts. Just to name a few, starting with the paper of Atiyah and Bott, unexpected links with mathematical physics were perceived, coming from the so-called Yang–Mills equations, where they prove that irreducible unitary representations realise the minimum, in the Morse theoretic sense, of the Yang–Mills functional. This led to a radically different differential geometric approach to these problems and vast ramifications in mathematical physics and 4-manifold topology, leading for instance to the deep work of Simon Donaldson.

In his papers of 1967 and 1968 ([8, 9]), Seshadri then proceeded to compactify the moduli spaces $M(n, d)^s$ by extending Mumford's Geometric Invariant Theory. The notion of a *semi-stable* bundle is a slight generalisation of that of a stable bundle and these under a special kind of equivalence provide the points needed to compactify the moduli space. The fundamental notions which had its origins in these papers, such as that of "S-equivalence" and the technique of proving that a bundle is semi-stable *if and only if* it happens to be GIT semi-stable in a suitable space, have become the standard tools in most constructions of compactifications found in the literature. Indeed, to be able to generalise the moduli constructions to fields of arbitrary characteristics, it was firstly essential that GIT be generalised to arbitrary characteristics and secondly, to be able to prove the properness of the moduli without using the compactness of the underlying topological space (what is now known as Langton's valuative criterion); this is achieved in [9].

In fact, David Mumford pointed out in his talk during Seshadri's sixtieth birthday celebrations that Seshadri's construction of the compactification of the moduli of stable bundles with all its conceptual complexity was a perfect representative example and a forerunner of all later GIT constructions of compactifications in a whole range of moduli problems.

In the mid-1960s, towards generalising GIT for fields of arbitrary characteristics, D Mumford made his conjecture on the equivalence of *geometric reductivity* and *reductivity*. Seshadri proved this conjecture in 1968 for the case of $GL(2)$ ([9]); earlier T Oda had proven this for a field of characteristic 2. Viewed from a geometric standpoint, the *projectivity* of the moduli space of semistable bundles *thought of as a GIT quotient* provides strong evidence for the validity of Mumford conjecture for $GL(n)$. Following this train of thought, Seshadri wrote his paper ([11]) on Quotient spaces in 1970 as an attempt to solve this conjecture using algebro-geometric methods. The paper was not quite successful in proving the Mumford conjecture, however it contained many fundamental ideas such as for instance what is now known as "Seshadri's ampleness criterion" and the so-called "Seshadri constant" which plays a key role in the classification of algebraic varieties. The conjecture which was finally settled by W Haboush in the mid-1970s using crucially the work of Steinberg in representation theory. But very recently, drawing on results of Sean Keal from a paper in 1998, Seshadri (in collaboration with P Sastry [17]) has given a more algebro-geometric proof of the Mumford Conjecture which he had envisaged in his paper ([11]) of 1970 on Quotient spaces.

Seshadri, inspired again by A Weil's "Généralisation des fonctions abéliennes", went on to define the notion of a *parabolic bundle* as the natural analogue for studying the bundle theoretic aspects of representation theory of more general Fuchsian groups (see [10]). His paper ([13]) (written in collaboration with V B Mehta) which interprets unitary representation of Fuchsian groups as parabolic semi-stable bundles, has had profound applications in the synthesis of mathematical physics and topology. In a sense, this work of Seshadri gave a final shape to the theme that had been envisaged in Weil's paper. Very recently, in a joint work ([19]), Balaji and Seshadri interpret homomorphisms of Fuchsian groups into maximal compact subgroups of semisimple algebraic groups in terms of torsors under Bruhat–Tits groups schemes which need not be semisimple. This is in striking contrast with the earlier results on parabolic vector bundles.

One of the difficult problems in the study of vector bundles on curves is that of classification of bundles on singular curves. Seshadri constructed a natural compactification of semistable vector bundles on irreducible nodal curves by adding torsion-free sheaves under a suitable equivalence. The key property of this construction was that it had good *specialisation* properties ([14]). This problem was not trivial even when the bundles were of rank 1 and this was studied in great depth by Oda and Seshadri (see [18] and [12]). Subsequently, in collaboration with D S Nagaraj ([15, 16]), Seshadri has made significant progress in the general problem of compactifications with "normal crossing singularities", generalising the work of Gieseker who had done it earlier for the rank 2 case. The problem of constructing projective moduli spaces of sheaves on nodal curves has many applications especially towards solving several topological questions on $M(n, d)$.

Seshadri's contributions to the field of representation theory and standard monomials has been dealt with in detail by an article by Seshadri himself in the second volume of his recently published collected papers.

We now turn to give a very brief account of his work on standard monomial theory much of which in its later developments was a collaboration with V Lakshmibai and C Musili. The modern standard monomial theory was initiated by C S Seshadri in the early 1970's which was a vast generalisation of the classical theory of Hodge for the Grassmannians. The broad aim of this theory was the construction of bases for the space of sections of line bundles on Schubert varieties which reflects the intrinsic geometry of the Schubert variety and the intricate combinatorics of the Weyl group. The theory has led to very fundamental developments in the fields of Representation theory, Geometry and Combinatorics. Following a series of basic papers written in collaboration with V Lakshmibai and being guided by careful analysis and a study of Schubert varieties for exceptional groups, Lakshmibai and Seshadri formulated the LS conjectures. The key point of the conjectures was that it gave an indexing of the SMT bases which implied a remarkable character formula now termed the LS character formula. There was a second aspect to these conjectures which constructed bases for the usual Demazure modules associated to the Schubert varieties. P Littelmann proved these conjectures by bringing in fresh inputs and new ideas from the theory of Quantum groups.

3. Seshadri's Contribution to Mathematics Education

The Chennai Mathematical Institute in its present form was founded in 1998 but its roots go back to 1989 when Seshadri founded a new institute, then called the School of Mathematics, SPIC Science Foundation. The Chennai Mathematical Institute (CMI) is a unique institution in India which attempts to integrate undergraduate education with research; it grew out of Seshadri's vision that higher learning can be only in an atmosphere of active research amidst the presence of masters in the subject. It was a brave venture in the face of extraordinary opposition and skepticism even from his very close friends and well-wishers. It was his dream to build a centre of learning which can compare itself with the great centres such as the Ecole Normale in Paris, the Oxford and Cambridge Universities in England and the Harvard University in the US. It opens up opportunities for the gifted students in India to learn in this unique academic atmosphere and also gives possibilities for the active researchers to participate in this experiment which one believes will leave an everlasting influence on the development of mathematics in India. It would not be an exaggeration to say that the Chennai Mathematical Institute is now rated as one of the best schools in the world for undergraduate studies in mathematics. This is indeed a first step in its stride and much still needs to be done to fulfill Seshadri's dream.

References

[1] C. S. Seshadri, Triviality of vector bundles over the affine space K, *Proc. Mat. Aca. Sci., USA* **44** (1958) 456–458.

[2] C. S. Seshadri, Many exposes in the Chevalley Seminar, Seminaire Chevalley, 3 Aunce (1958–1959).

[3] C. S. Seshadri, Variete de Picard d'une Variete complete, *Annali di Mat. Italy IV* **LVII** (1962) 117–142.

[4] C. S. Seshadri, Some results on the quotient space by an algebraic group of automorphism, *Math. Annalen* **149** (1963) 286–301.

[5] C. S. Seshadri, Quotient space by an Abelian variety, *Math. Annalen* **152** (1963) 185–194.

[6] M. S. Narasimhan and C. S. Seshadri, Holomorphic vector bundles on a compact Riemann surface, *Math. Ann.* **155** (1964) 69–80.

[7] M. S. Narasimhan and C. S. Seshadri, Stable bundles and unitary bundles on a compact Riemann surface, *Annals of Math.* **82** (1965) 540–567.

[8] C. S. Seshadri, Space of unitary vector bundles on a compact Riemann surface, *Annals of Math.* **85** (1967) 303–335.

[9] C. S. Seshadri, Mumford's conjecture for $GL(2)$ and applications, *Proc. Intern. Colloquium on Algebraic Geometry, Bombay* (1968) 347–371.

[10] C. S. Seshadri, Moduli of π-vector bundles over an algebraic curve, *Proc. C.I.M.E. Session, Varenna* (1969).

[11] C. S. Seshadri, Quotient spaces module reductive algebraic groups, *Ann. Math.* **95**(3) (1972) 511–556.

[12] T. Oda and C. S. Seshadri, Compactifications of the generalised Jacobian variety, *Transactions of the A.M.S.* **253** (1979) 190.

[13] V. B. Mehta and C. S. Seshadri, Moduli of vector bundles on curves with parabolic structures, *Math. Ann.* **228** (1980) 205–239.

[14] C. S. Seshadri, Vector bundles on curves, Lectures at the Ecole Normale Superieure (1980) — Asterisque, 96.

[15] D. S. Nagaraj and C. S. Seshadri, Degenerations of the moduli spaces of vector bundles on curves I, *Proc. Indian Acad. Sci. (Math. Sci.)* **107** (1997) 101–137.

[16] D. S. Nagaraj and C. S. Seshadri, Degenerations of the moduli spaces of vector bundles on curves II (Generalized Gieseker moduli spaces), *Proc. Indian Acad. Sci. (Math. Sci.)* **109**(2) (1999) 165–201.

[17] P. Sastry and C. S. Seshadri, Geometric reductivity — A quotient space approach to appear in *Journal Ramanujan Mathematical Society* (2011).

[18] C. D'Souza, Compactication of generalised Jacobians, *Proc. Indian Acad. Sci. Sect. A Math. Sci.* **88**(5) (1979) 419–457.

[19] V. Balaji and C. S. Seshadri, Moduli of parahoric $\mathcal{G}$-torsors on a compact Riemann surface, preprint (2011).

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