Distributive Lattices and Coherence in Homological Algebra

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Abstract. This article is about coherence in homological algebra, and only needs the elementary theory of abelian groups and lattices. Its results are developed in a recent book to analyse spectral sequences — an important tool of homological algebra with applications in many branches of mathematics.

0. Introduction

Spectral sequences, one of the main tools of homological algebra (see [12, 8, 11, 17, 19, 13]), find applications in many branches of mathematics, from algebraic topology to algebraic geometry, differential geometry and partial differential equations, thence to physics through the C-spectral sequence of a PDE [18, 15] and control theory [9].

This expository article is about coherence in homological algebra. The main applications, developed in a recent book [7], deal with spectral sequences, but this exposition only needs the elementary theory of abelian groups and lattices.

In fact, various parts of homological algebra are based on "induction on subquotients" (i.e. quotients of subgroups, or — equivalently — subgroups of quotients). However, the coherence of this procedure of induction leads to serious problems, that are often overlooked.

Problems may already arise in the simplest situation, *canonical isomorphisms* between subquotients of the same object (induced by the identity of the latter): first, such isomorphisms need not be closed under composition; second, if we extend them in this sense the result need not be uniquely determined (as shown in Sec. 3). Yet, such isomorphisms are frequently used when working with complicated systems, in particular those that give rise to spectral sequences.

The solution to this coherence problem depends on the distributivity of the lattices of subgroups generated by the system that we are studying. We prove in Sec. 6 the following theorem:

Given a sublattice X of the (modular) lattice of all subgroups of an abelian group A, let us consider the subquotients M/N of A with M, N belonging to X. Then the canonical isomorphisms among these subquotients are closed under composition (and form a coherent system) if and only if the lattice X is distributive.

This is an elementary form of our "Coherence theorem for homological algebra". A more complete form of the theorem, sketched in Sec. 8, can be found in the book [7] (see also [4]–[6]).

These works prove that the following systems are "distributive", i.e. they generate distributive lattices of subgroups and *their coherence automatically holds*:

– bifiltered object,
– sequence of morphisms,
– filtered chain complex,
– double complex,
– Massey's exact couple [12],
– Eilenberg's exact system [2].

The same property of distributivity also permits representations of these structures by means of *lattices of subsets*; this yields a precise foundation for the heuristic tool of Zeeman diagrams [20, 8], as universal models of spectral sequences.

On the other hand, the *bifiltered* chain complex is *not* distributive, and we show in [7] a strong form of inconsistency in this system, that can lead to gross errors if the interaction of the two spectral sequences of the complex is explored further than it is normally done.

The symbol $\subset$ always denotes *weak* inclusion (of subsets, subgroups, etc.).

1. Subquotients and Regular Induction

For the sake of simplicity, we work in the category Ab of abelian groups, but everything can be extended to abelian categories and even further (see Sec. 8).

A subquotient $S = M/N$ of an abelian group A is a quotient of a subgroup (M) of A, or equivalently a subgroup of a quotient (A/N). It is determined by two subgroups $N \subset M$ of A, via a commutative square that is bicartesian, i.e. pullback and pushout:

$$\begin{array}{ccc} M & \xrightarrow{m} & A \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ S & \xrightarrow[n]{} & A/N \end{array} \tag{1}$$

The prime example, of course, is the homology subquotient $H = \operatorname{Ker}\partial/\operatorname{Im}\partial$ of a differential group (A, ∂); more complex examples come from the terms of spectral sequences.

A homomorphism $f\colon A \to B$ is given. If M and H are subgroups of A and B respectively, and $f(M) \subset H$, we have a commutative diagram with short exact rows, and two induced homomorphisms

$$\begin{array}{ccccc} M & \xrightarrow{m} & A & \xrightarrow{p} & A/M \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} \\ H & \xrightarrow[h]{} & B & \xrightarrow[u]{} & B/H \end{array} \tag{2}$$

More generally, given two subquotients M/N of A and H/K of B, suppose that:

$$f(M) \subset H \quad \text{and} \quad f(N) \subset K. \tag{3}$$

Then, we have a *regularly induced* homomorphism $g\colon M/N \to H/K$. In fact, one can form the diagram below, by applying (2) in two ways

$$\begin{array}{ccccc} M & \xrightarrow{m} & A & \xrightarrow{q} & A/N \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} \\ H & \xrightarrow[h]{} & B & \xrightarrow[v]{} & B/K \end{array} \tag{4}$$

Then we get a new commutative diagram, and the homomorphism g, by factorisation of the rows of the previous diagram through their images

$$\begin{array}{ccccc} M & \xrightarrow{q'} & M/N & \xrightarrow{m'} & A/N \\ {\scriptstyle f'}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f''} \\ H & \xrightarrow[v']{} & H/K & \xrightarrow[h']{} & B/K \end{array} \tag{5}$$

Regular induction is (obviously) preserved by composition. But it is too restricted a notion.

2. Canonical Isomorphisms

We now use the category RelAb of (additive) *relations*, or (additive) *correspondences*, of abelian groups (cf. Mac Lane [10]).

A relation $a\colon A \to B$ is a subgroup of the direct sum $A \oplus B$. It can be viewed as a "partially defined, multi-valued homomorphism", that sends an element $x \in A$ to the subset $\{y \in B \mid (x, y) \in a\}$ of B. The composite ba, with $b\colon B \to C$, is

$$\{(x, z) \in A \oplus C \mid \exists y \in B\colon (x, y) \in a,\ (y, z) \in b\}.$$

The converse relation of $a\colon A \to B$ is obtained by reversing pairs, and written as $a^\sharp\colon B \to A$. This involution is regular in the sense of von Neumann, i.e. $aa^\sharp a = a$, for all relations a. Therefore a *monorelation*, i.e. a monomorphism in the category RelAb, is characterised by the condition $a^\sharp a = 1$, and an *epirelation* by the condition $aa^\sharp = 1$.

The category Ab is embedded in RelAb, identifying a homomorphism with its graph. This embedding preserves monomorphisms and epimorphisms (but we shall see that a monorelation is more general than an injective homomorphism). *Isomorphisms are the same*, in these categories.

Let us come back to the bicartesian square making S into a subquotient M/N of the abelian group A

$$\begin{array}{ccc} M & \xrightarrow{m} & A \\ {\scriptstyle p}\downarrow & \nearrow{\scriptstyle s} & \downarrow{\scriptstyle q} \\ S & \xrightarrow[n]{} & A/N \end{array} \tag{6}$$

This square determines *one relation* $s = mp^\sharp = q^\sharp n\colon S \to A$, that sends the class $[x] \in M/N$ to all the elements of the lateral $x + N$ in A. It is actually a monorelation (since $s^\sharp s = \mathrm{id}(S)$) and all monorelations with values in A are of this type, up to isomorphism. The *subquotients* of the abelian group A amount thus to the *subobjects of A in* RelAb.

RelAb makes possible to consider a more general notion of induction on subquotients, as defined in [10]. Given a relation $a\colon A \to B$ and two subquotients $s\colon S \to A$, $t\colon T \to B$, we say that a *induces from s to t* the relation

$$t^\sharp as\colon S \to T. \tag{7}$$

In particular, if a is a homomorphism with a regularly induced homomorphism $S \to T$, the latter coincides with $t^\sharp as$.

If s, t are subquotients of the same object A, the relation $u = t^\sharp s\colon S \to T$ induced by the identity of A will be called the *canonical relation* from s to t; and a *canonical isomorphism* if it is an isomorphism (of RelAb or Ab, equivalently). The isomorphism u need not be regularly induced.

Writing the subquotient s as H/K, and t as H'/K', it is easy to verify the following properties of the canonical relation $u = t^\sharp s \colon H/K \to H'/K'$:

(a) u is everywhere defined $\Leftrightarrow$ $H \subset H' \vee K$,
(a*) u has total values $\Leftrightarrow$ $H' \subset H \vee K'$,
(b) u has a null annihilator $\Leftrightarrow$ $H \wedge K' \subset K$,
(b*) u is single-valued $\Leftrightarrow$ $H' \wedge K \subset K'$,
(c) u is an isomorphism $\Leftrightarrow$ $(H \vee K' = H' \vee K,\ H \wedge K' = H' \wedge K)$.

It follows that

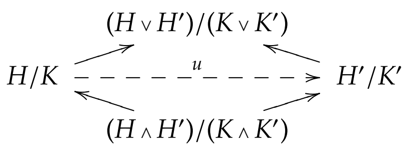
(d) u is a regularly induced isomorphism $\Leftrightarrow$ $(K = H \wedge K',\ H' = H \vee K')$,

which shows that a regularly induced isomorphism is the same as a second-type Noether isomorphism

$$H/(H \wedge K') \to (H \vee K')/K'. \tag{8}$$

We write $H/K\,\Phi\,H'/K'$ the property expressed in (c). It is obviously reflexive and symmetric, but *not transitive in general*, as shown below.

It is easy to see that, if $H/K\,\Phi\,H'/K'$, there is a commutative diagram of canonical isomorphisms (between Φ-related subquotients of A)



(9)

Note that the solid arrows are *regularly induced* (Noether) isomorphisms; this is important, because regular induction is always respected by composition.

3. A Case of Incoherence

The following examples show some instances of *inconsistency of induction on subquotients*: first, canonical isomorphisms need not be closed under composition; second, if we extend them in this sense the result need not be uniquely determined.

As in Mac Lane's book [10], our examples of inconsistency are based on the lattice $L(A)$ of subgroups of $A = \mathbb{Z} \oplus \mathbb{Z}$, and more particularly on the (*non-distributive*) triple formed of the diagonal Δ and two of its complements, the subgroups A_1 and A_2

$$\begin{aligned} &A_1 = \mathbb{Z} \oplus 0, && A_2 = 0 \oplus \mathbb{Z}, \\ &A_i \vee \Delta = A, && A_i \wedge \Delta = 0. \end{aligned} \tag{10}$$

We thus have the subquotients $m_i \colon A_i \to A$ and $s = p^\sharp \colon A/\Delta \to A$.

(a) The identity of A induces two canonical isomorphisms $u_i = pm_i \colon A_i \to A/\Delta$ (regularly induced Noether isomorphisms, by (10)), and a canonical isomorphism $u_2^{-1} \colon A/\Delta \to A_2$ (that is *not* regularly induced).

Then, the composed isomorphism $w = u_2^{-1}u_1 \colon A_1 \to A_2$ *is not canonical*. Indeed:

$$\begin{aligned} &w \colon A_1 \to A/\Delta \to A_2, \\ &(x, 0) \mapsto [(x, 0)] = [(0, -x)] \mapsto (0, -x), \end{aligned} \tag{11}$$

while the canonical relation $m_2^\sharp.m_1 \colon A_1 \to A_2$ has graph $\{(0,0)\}$.

(b) Using the subgroup $\Delta' = \{(x, -x) \mid x \in \mathbb{Z}\}$ instead of the diagonal Δ, we get the *opposite* composed isomorphism from A_1 to A_2

$$\begin{aligned} &A_1 \to A/\Delta' \to A_2, \\ &(x, 0) \mapsto [(x, 0)] = [(0, x)] \mapsto (0, x). \end{aligned} \tag{12}$$

This shows that a composite $A_1 \to A_2$ of canonical isomorphisms between subquotients of $\mathbb{Z}^2$ *is not determined*.

Now, a change of sign can be quite important, in homological algebra and algebraic topology. For instance, it is the case in the usual argument proving that "even-dimensional spheres cannot be combed": if the sphere $\mathbb{S}^n$ has a non-null vector field, then its antipodal map $t \colon \mathbb{S}^n \to \mathbb{S}^n$ is homotopic to the identity, and the degree $(-1)^{n+1}$ of t must be 1. The conclusion cannot be obtained if we only know the induced homomorphism $t_{*n} \colon H_n(\mathbb{S}^n) \to H_n(\mathbb{S}^n)$ up to sign change.

4. Coherent Systems of Isomorphisms

Let X be a sublattice of the (modular) lattice $L(A)$ of subgroups of the abelian group A; we always assume that X contains the least and greatest elements of $L(A)$. We are interested in the set $\hat{X}$ of all the subquotients of A with numerator and denominator in X, whose coherence is discussed below.

Plainly, the set $\hat{X}$ can be identified with the set X_2 of decreasing pairs (numerator, denominator) of X, where the relation $(x, y)\,\Phi\,(x', y')$ is expressed by the following two equivalent conditions:

(a) $x \vee y' = x' \vee y,\quad x \wedge y' = x' \wedge y,$
(b) $x \leqslant x' \vee y,\ x' \leqslant x \vee y',\quad x \wedge y' \leqslant y,\ x' \wedge y \leqslant y'.$

For a system Σ of subquotients of A (usually of the previous form), we are interested in the following equivalent properties

(i) whenever $u \colon S \to S'$ and $v \colon S' \to S''$ are induced isomorphisms between elements of the system, the composed isomorphism vu coincides with the canonical relation $S \to S''$,

(ii) the relation Φ is an equivalence relation among the subquotients of Σ, and all diagrams of canonical isomorphisms between them commute.

When this holds we say that Σ is a *coherent system of subquotients* of A. We also express this fact saying that the canonical isos among all $S \in \Sigma$ *are closed under composition*, or *form a coherent system of isomorphisms*. (Since Mac Lane's paper [11], a coherence theorem in category theory states that, under suitable assumptions, all the diagrams of a given type commute.)

When such a system has been fixed (e.g. using the Coherence theorem below) we shall express the (equivalence!) relation $S \,\Phi\, S'$ of Σ by saying that the subquotients S and S' are *canonically isomorphic* (within Σ). But expressing in this way the relation Φ when transitivity does not hold is misleading and should be carefully avoided.

We have seen that the whole system of subquotients of $\mathbb{Z}^2$ is not coherent (Sec. 3); the same holds for any abelian group $A \oplus A$, where A is not trivial.

Even when a set $\Sigma = \hat{X}$ is coherent, one should not expect that all the induced *homo*morphisms (or even less relations) be closed under composition. In fact, the composite of the canonical homomorphisms

$$A/0 \to A/A \to 0/0 \to A/0$$

is null, while the canonical homomorphism $A/0 \to A/0$ is the identity (and all these subquotients necessarily belong to $\hat{X}$).

5. Lemma

Let X be a modular lattice. The following conditions are equivalent:

(i) *the lattice X is distributive,*

(ii) *the relation $(x, y)\,\Phi\,(x', y')$ defined above on the set X_2 of decreasing pairs of X is an equivalence relation.*

Proof. Let X be distributive, and assume that $(x, y)\,\Phi\,(x', y')\,\Phi\,(x'', y'')$. Then:

$$x = (x' \vee y) \wedge x = (x'' \vee y' \vee y) \wedge x$$
$$\leqslant x'' \vee (y' \wedge x) \vee y = x'' \vee (y \wedge x') \vee y = x'' \vee y.$$

The other three inequalities of $(x, y)\,\Phi\,(x'', y'')$, in form (b) of Sec. 4, are proved in a similar way.

Conversely, suppose that the relation Φ is transitive. Let $M = \{m', x, y, z, m''\}$ be a sublattice of X, where the meet (resp. join) of any two elements out of x, y, z is m' (resp. m''). Then we have $(x, m')\,\Phi\,(m'', y)\,\Phi\,(z, m')$, whence $(x, m')\,\Phi\,(z, m')$ and $x = z$.

In other words, the modular lattice X cannot have a sublattice M as above, formed of five *distinct* elements; by a well-known theorem ([1], II.8, Theorem 13), X must be distributive.

6. Coherence Theorem of Homological Algebra (Reduced Form)

Theorem. *Let X be a sublattice of the lattice $L(A)$ of subgroups of the abelian group A. Then the following conditions are equivalent*:

(i) *the lattice X is distributive,*

(ii) *the family $\hat{X}$ is coherent* (i.e. *the canonical isomorphisms among subquotients of A with numerator and denominator belonging to X are closed under composition*).

Proof. If (ii) holds, the relation Φ is transitive in $\hat{X}$ (or equivalently in X_2) and X is distributive, by the previous lemma.

Conversely, let us assume that X is distributive, and consider two canonical isomorphisms between three subquotients

$$u\colon H/K \to H'/K', \quad v\colon H'/K' \to H''/K''. \tag{13}$$

We must prove that the composite vu is the canonical relation $w\colon H/K \to H''/K''$. By Lemma 5, these three subquotients are Φ-equivalent.

Let us write

$$S_1 = (H \wedge H')/(K \wedge K'),$$
$$S_2 = (H' \wedge H'')/(K' \wedge K''),$$
$$S_0 = (H \wedge H' \wedge H'')/(K \wedge K' \wedge K'').$$

By Sec. 2, we can form the following commutative diagram, where all subquotients are Φ-equivalent, and the solid arrows are *regularly* induced by $\mathrm{id}(A)$

$$\begin{array}{ccccc} H/K & \overset{u}{\dashrightarrow} & H'/K' & \overset{v}{\dashrightarrow} & H''/K'' \\ \uparrow & \nearrow & & \nwarrow & \uparrow \\ S_1 & & & & S_2 \\ & \nwarrow & & \nearrow & \\ & & S_0 & & \end{array} \tag{14}$$

But we can also form a second commutative diagram with regularly induced solid arrows

$$\begin{array}{ccccc} H/K & & \overset{w}{\dashrightarrow} & & H''/K'' \\ \uparrow & \nwarrow & & \nearrow & \uparrow \\ S_1 & & (H \wedge H'')/(K \wedge K'') & & S_2 \\ & \nwarrow & \uparrow & \nearrow & \\ & & S_0 & & \end{array} \tag{15}$$

Since the four solid arrows of the "boundary" of these two diagrams coincide, the thesis follows: $vu = w$.

7. Filtered Chain Complexes

Let us now consider one of the most usual structures giving rise to a spectral sequence, a *filtered chain complex* A_* of abelian groups, with (canonically) *bounded* filtration

$$A_* = ((A_n), (\partial_n), (F_pA_n)). \tag{16}$$

This is a chain complex of abelian groups

$$\dots A_n \xrightarrow{\partial_n} A_{n-1} \to \dots \to A_1 \xrightarrow{\partial_1} A_0$$
$$(\partial_n\partial_{n+1} = 0),$$

where each component A_n has a filtration of length $n+1$, consistently with the differentials:

$$0 \subset F_0A_n \subset \dots F_pA_n \subset \dots F_nA_n = A_n,$$
$$\partial_{n+1}(F_pA_{n+1}) \subset F_pA_n. \tag{17}$$

On each component A_n the filtrations of A_{n+1} and A_{n-1} produce a second finite filtration (of length $2n+3$), by direct and inverse images along the differentials (while the other components have a trivial effect)

$$0 \subset \partial_{n+1}(F_0A_{n+1}) \subset \dots \partial_{n+1}(F_{n+1}A_{n+1})$$
$$= \operatorname{Im}\partial_{n+1} \subset \operatorname{Ker}\partial_n = \partial_n^{-1}(0) \subset \tag{18}$$
$$\partial_n^{-1}(F_0A_{n-1}) \subset \dots \partial_n^{-1}(F_{n-1}A_{n-1}) = A_n.$$

By a well-known Birkhoff theorem on the free modular lattice generated by two chains ([Bi], III.7, Theorem 9), the two filtrations generate a finite, *distributive* lattice of subgroups of A_n, that can be represented as (a quotient of) a lattice of subsets of the plane. (Notice the crucial role played here by the condition $\partial\partial = 0$: without that, the lattice generated by the data would not be distributive.)

In particular, F_pA_n has a filtration of *relative cycles* and *relative boundaries*, that is the "trace" of the second filtration (18) (with $n = p + q$)

$$Z^r_{pq}(A_*) = F_pA_n \wedge \partial^{-1}(F_{p-r}A_{n-1}),$$
$$B^r_{pq}(A_*) = F_pA_n \wedge \partial(F_{p+r}A_{n+1}). \tag{19}$$

Now, the term E^r_{pq}, of the spectral sequence of A_* is usually defined as a subquotient of A_n (with $n = p + q$), by one of the following "equivalent" formulas:

$$Z^r_{pq}/(Z^{r-1}_{p-1,q+1} \vee B^{r-1}_{pq}), \tag{20}$$

$$(Z^r_{pq} \vee F_{p-1}A_n)/(B^{r-1}_{pq} \vee F_{p-1}A_n), \tag{21}$$

that are linked by a canonical isomorphism, regularly induced from the first to the second subquotient.

The first expression is used, for instance, in Hilton–Wylie [8, Section 10.3], and Spanier [17, 9.1]. The second is used in Mac Lane's "Homology" [10, XI.3]. Weibel [19] uses both, in Sec. 5.4 (with a different notation).

And indeed, no problem can *here* arise from using different formulas linked by canonical isomorphisms, *because of the distributivity of the system*. But this is no longer true in a *non-distributive* system like the bifiltered chain complex [7], if we let its spectral sequences (derived from the two filtrations) interact.

8. The Full Coherence Theorem

We end by mentioning, without proof, a more complete form of our coherence theorem.

The proof can be found in the book [7], with various other equivalent conditions and many applications to the theory of spectral sequences.

The required setting is an extension of abelian categories. A *p-exact* category, i.e. an exact category in the sense of Puppe and Mitchell [16, 14, 3], is a category with a zero object, where every map factorises as a cokernel (of some morphism) followed by a kernel. The setting is selfdual, and the existence of cartesian products is not assumed.

A *p-exact* category E is said to be *distributive* if all its lattices of subobjects are distributive. The main example is the category $\mathcal{I}$ of sets and partial bijections, where every small distributive *p-exact* category can be exactly embedded.

A non-trivial abelian category cannot be distributive; but all *p-exact* categories, including the abelian ones, have a *distributive expansion* to which the theorem below applies.

Coherence theorem of homological algebra

For a p-exact category E, *the following conditions are equivalent*:

(i) *canonical isomorphisms between subquotients of the same object are closed under composition*,

(ii) *induced isomorphisms between subquotients* (*induced by arbitrary homomorphisms, or even by relations*) *are preserved by composition*,

(iii) E *is* distributive,

(iv) *the category of relations* $\mathsf{Rel}\mathsf{E}$ *is* orthodox (i.e. *its idempotent endomorphisms are closed under composition*).

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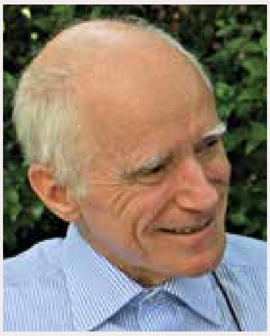

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